Quantum noise in statistical physics

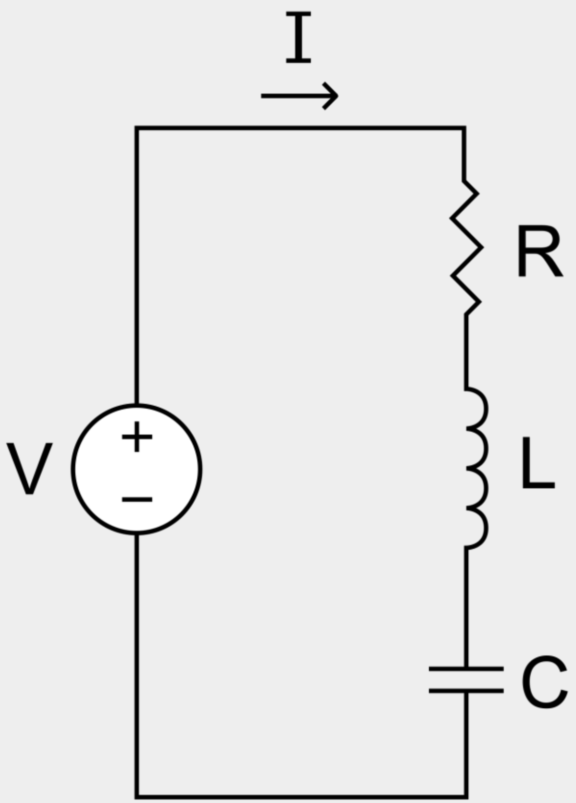
Nick Rivera

AEP/PHYS 4230/5230: Statistical Thermodynamics

Today: mostly qualitative and fun ☺

How to describe a damped system at $T=0$?

Johnson noise in the quantum limit



Consider the case without the resistor. The LC circuit is a harmonic oscillator:

$$H = \frac{P^2}{2L} + \frac{Q^2}{2C} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

with charge operator:

$$Q = \sqrt{\frac{\hbar}{2L\omega}}(a + a^\dagger)$$

Image source: Wikipedia

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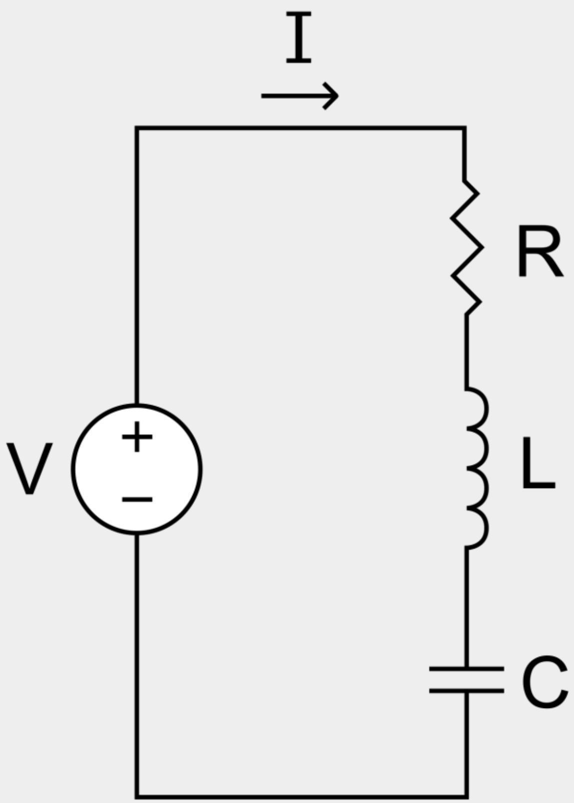

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The average energy is given by:

$$\langle E \rangle = \hbar\omega\left(\frac{1}{e^{\beta\hbar\omega} - 1} + \frac{1}{2}\right) = \frac{\hbar\omega}{2}\coth\left(\frac{\beta\hbar\omega}{2}\right)$$

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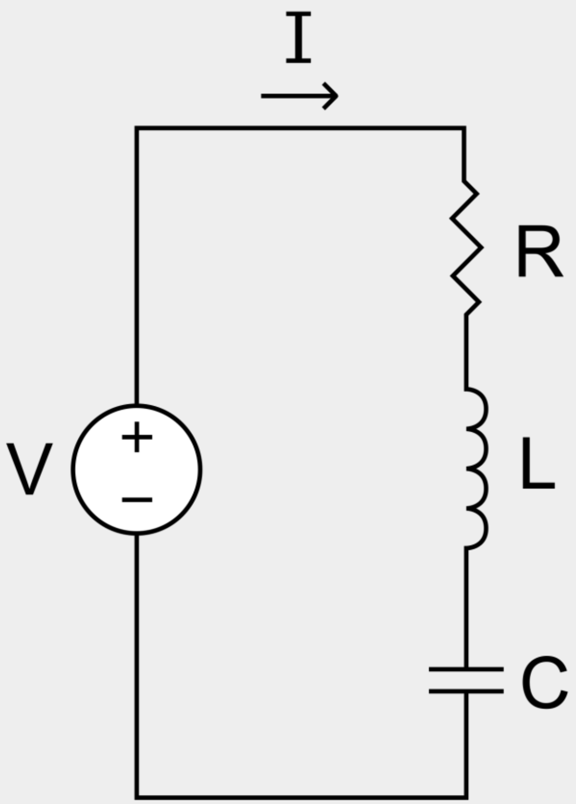

Image source: Wikipedia

The mean square charge follows from:

$$\left\langle \frac{Q^2}{2C} \right\rangle = \frac{1}{2}\langle E \rangle \implies \langle Q^2 \rangle = \frac{\hbar\omega C}{2} \coth\left(\frac{1}{2}\beta\hbar\omega\right)$$

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* This is approximate and only works well if the circuit is underdamped.

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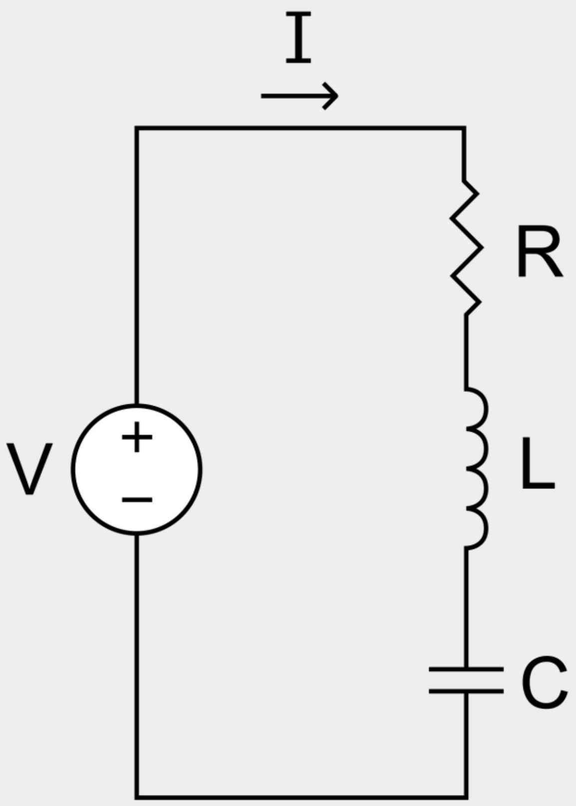

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In a damped RLC circuit system in thermal equilibrium, this variance must be as above, suggesting that we make the replacement*:

$$k_B T \to \frac{\hbar\omega}{2} \coth\left(\frac{1}{2}\beta\hbar\omega\right)$$

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Odd behavior of the quantum PSD

The PSD of the charge on the capacitor becomes

$$S_Q(\omega) = \frac{\hbar\omega R \coth(\frac{1}{2}\beta\hbar\omega)}{(C^{-1} - L\omega^2)^2 + \omega^2 R^2}$$

At zero temperature, this is nonzero, unlike in the classical Langevin treatment! Does the interpretation of the PSD as energy density carry through? Is there energy to be extracted from this circuit?

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No! *Circuit is in its ground state*.

Zero-point motion

Let's go back to mechanical oscillators which are mathematically equivalent but easier to think about. Consider the *ground state* of the harmonic oscillator.

$$|\psi(x)|^2 = \frac{e^{-x^2/2\sigma_x^2}}{(2\pi\sigma_x^2)^{1/2}} \implies \langle x \rangle = 0 \text{ and } (\Delta x)^2 = \frac{\hbar}{2m\omega}.$$

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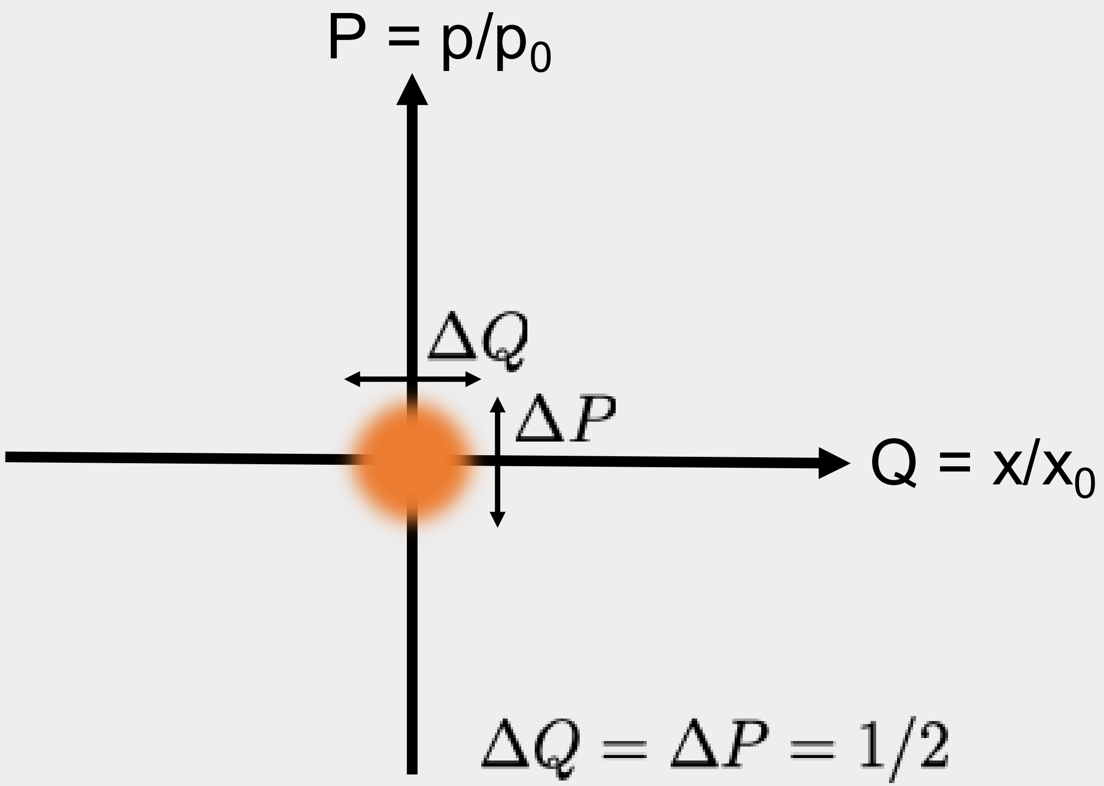
A finite variance means residual displacement and velocity, we call this "*zero-point motion*" because it occurs at **T = 0**. Consequence of $\Delta x \Delta p \geq \hbar/2$

Phase-space representation



In the phase space picture, the ground state of a harmonic oscillator can be seen almost like a classical random process of zero mean position and momentum but with Gaussian uncertainties in each.

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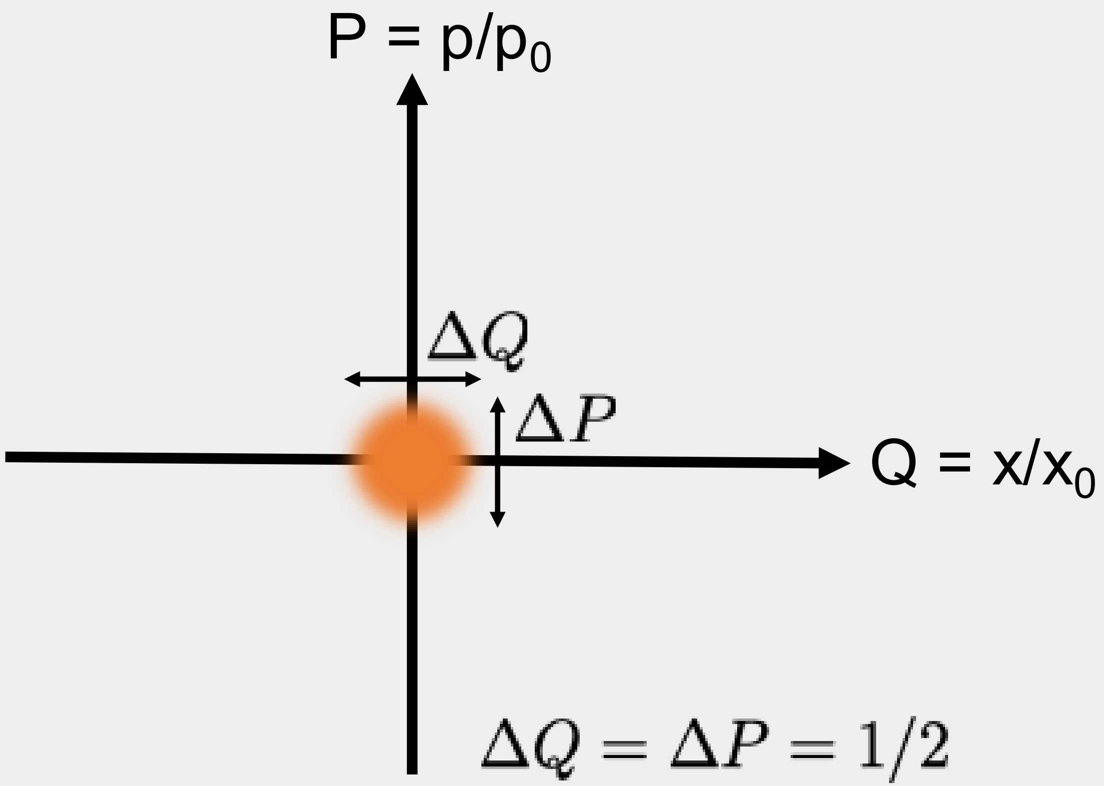

In the phase space picture, the ground state of a harmonic oscillator can be seen almost like a classical random process of zero mean position and momentum but with Gaussian uncertainties in each.

At the same time, it cannot *exactly* be the same as there is no *extractable energy in this system.*

Manifestations of zero-point motion

Parametric oscillator

Consider a mass on a spring where the spring constant *depends on time*:

$$m\ddot{x} + \omega_0^2(1 + \epsilon\cos(2\omega t))x = 0$$

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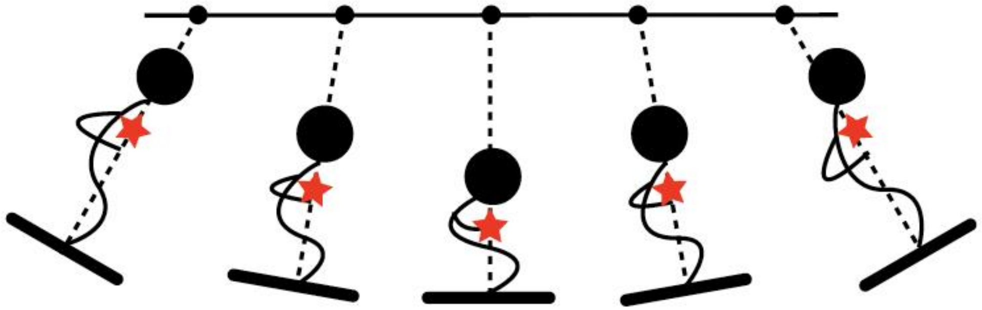

FIG. 1 (color online). Parametric amplification of pendulum motion by a child standing on a swing. The amplification is driven by changing the center of mass (stars), and thus effective length, of the pendulum at twice the frequency of the unperturbed swing.

Image source: American Physical Society

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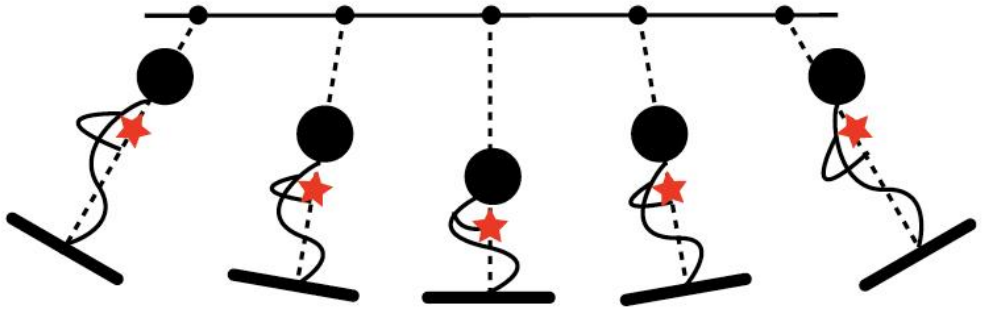

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$$x(t) = x(0)e^{\mu t}$$

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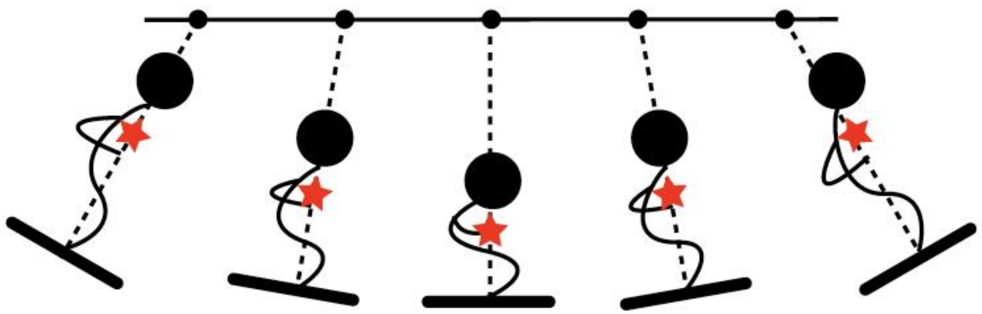

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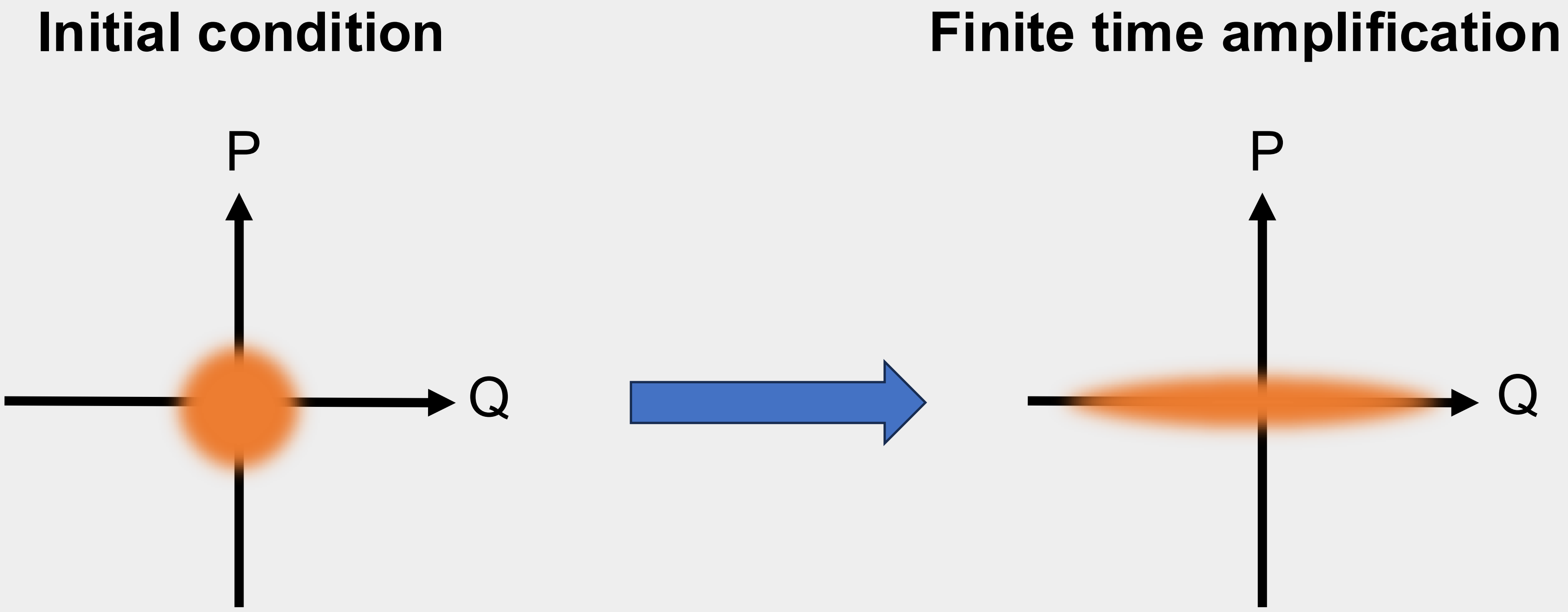
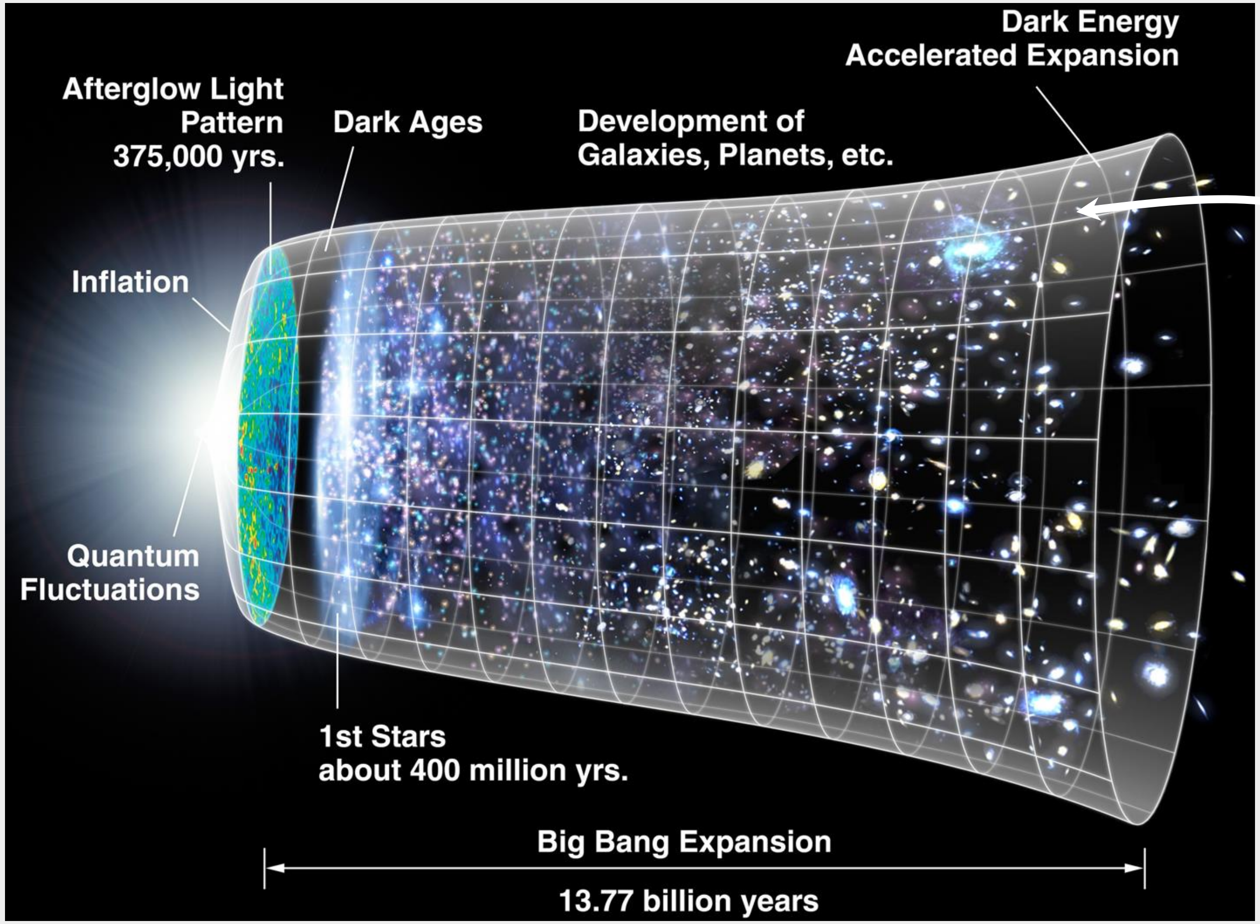
What happens if x(0) = 0, p(0) = 0?

Quantum noise gets amplified



Quantum noise in the early Universe



Large scale structure is suggested to emerge from a type of exponential amplification of small quantum fluctuations in the density of matter, leading to large-scale structure.

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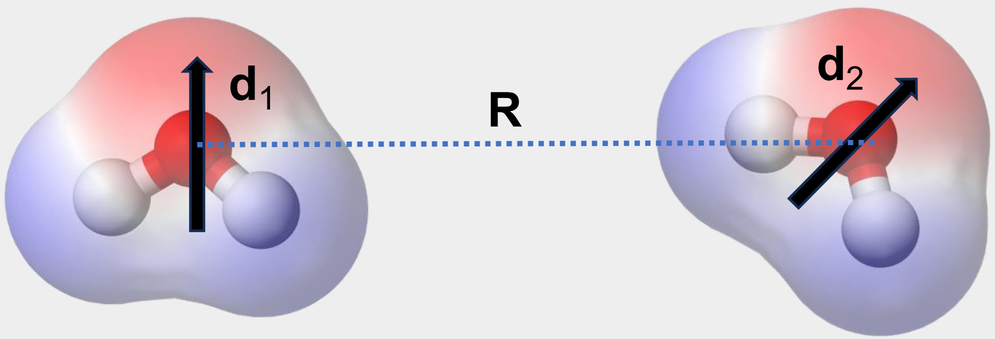
The van der Waals interaction

Electric dipole interactions

Consider two permament dipoles of moments $\mathbf{d}_1$ and $\mathbf{d}_2$ with a relative displacement of $\mathbf{R}$. There is an interaction potential between them:

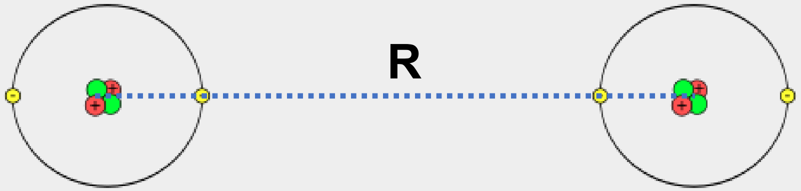


$$U = \frac{\mathbf{d}_1 \cdot \mathbf{d}_2 - 3(\mathbf{d}_1 \cdot \hat{R})(\mathbf{d}_2 \cdot \hat{R})}{4\pi\epsilon_0 R^3}$$

Image adapted from Wikipedia

Two neutral atoms: do they interact?

Consider two neutral nonpolar atoms like Helium.



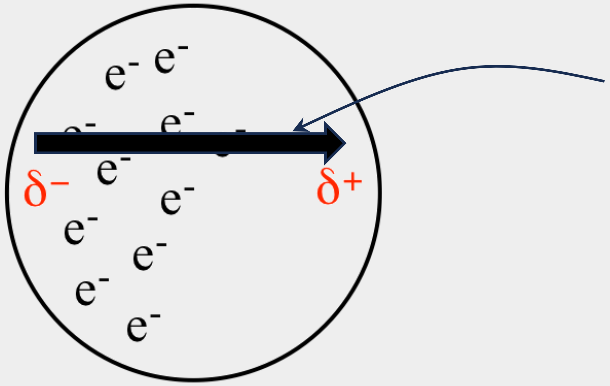
In this case, the mean dipole of each atom is zero, and so there is no dipole interaction. **Classically, they should not interact.**

Image adapted from Wikipedia

Van der Waals interaction

Each atom has a fluctuating dipole. For an atom in its ground state, similar to the ground state of many symmetric systems (particle in a box, harmonic oscillator, hydrogen atom), we have: $\langle \mathbf{r} \rangle = 0, \langle x^2 \rangle = (\Delta x)^2 \neq 0.$

This can be seen as a fluctuating dipole of magnitude: $q\sqrt{(\Delta r)^2} = \sqrt{3}q\Delta x$

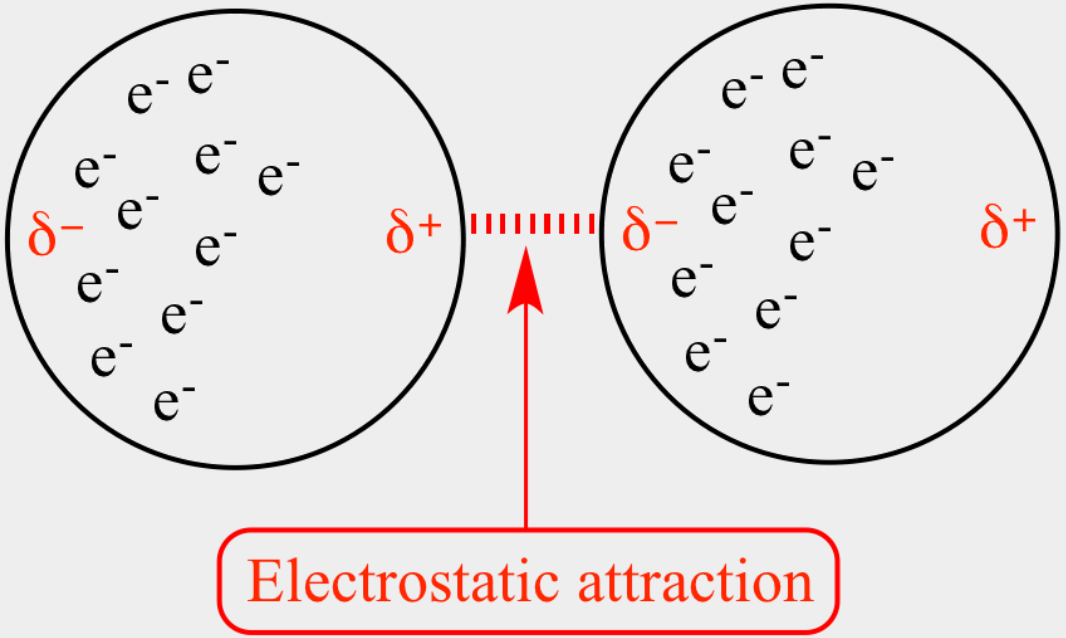


Instantaneous dipole of magnitude qΔx

Image adapted from: UCLA Chemistry dept.

Van der Waals interaction

When atoms are close together?



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Van der Waals interaction

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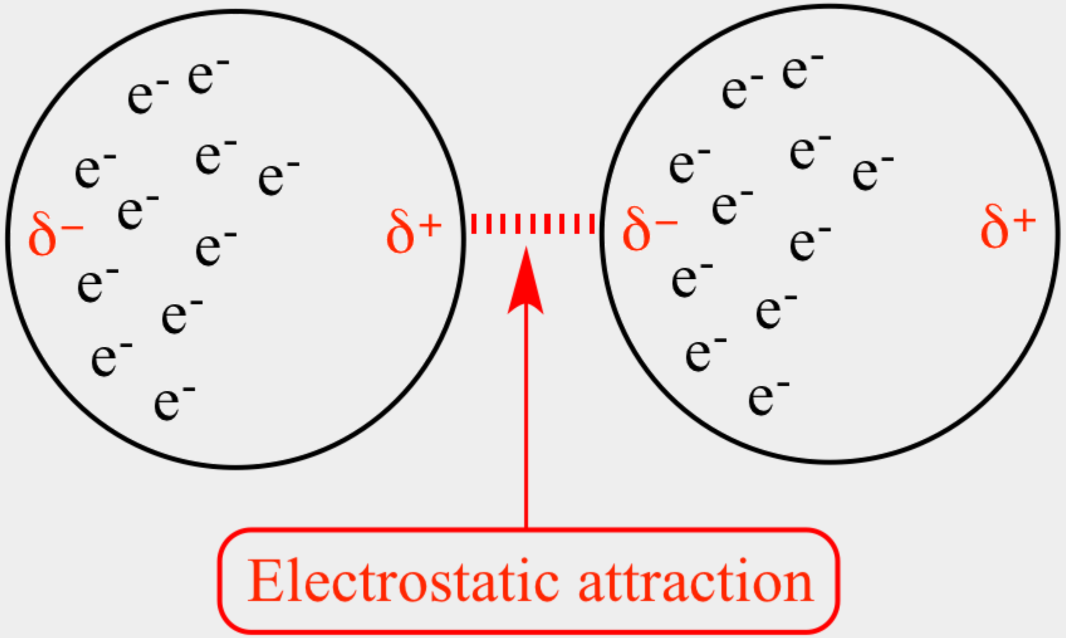

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Estimating magnitude of vdW

$$\langle U \rangle \sim -\frac{1}{R^3}\left(e\Delta x \times \alpha \frac{e\Delta x}{R^3}\right) \sim -\frac{C_6}{R^6}$$

Fluctuating dipole of atom 1

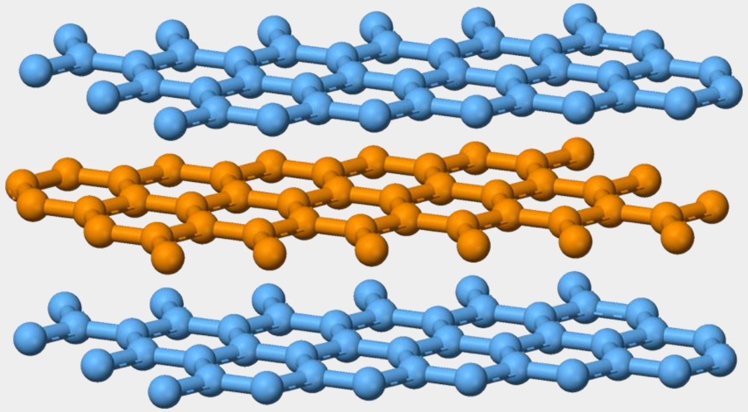
Induced dipole of atom 2

Very strong attractive force when particles are close by!

Of geckos and graphite

Image source: Wikipedia

Of geckos and graphite



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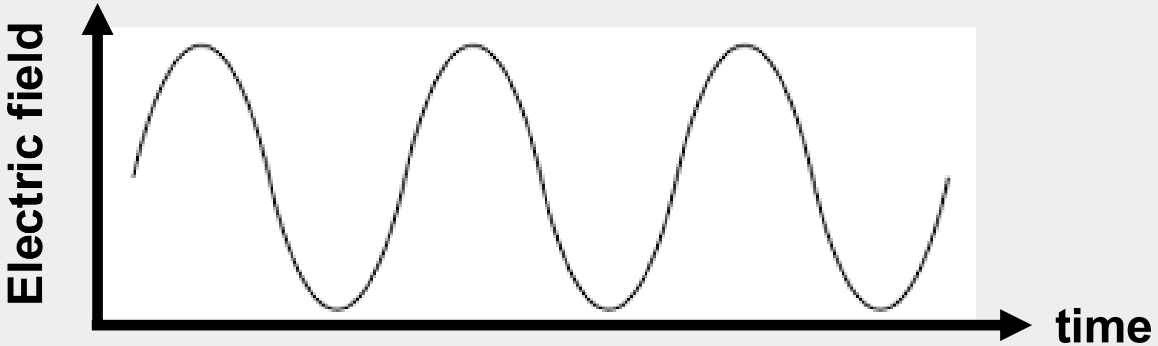
Shot noise of light and gravitational wave interferometry

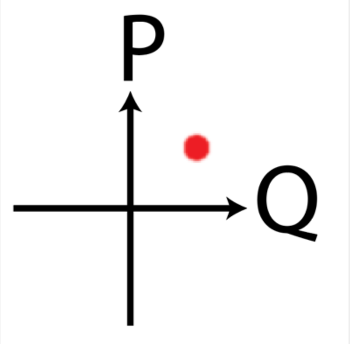
Quantum fluctuations of light



$$E(t), B(t) \rightarrow Q(t), P(t)$$

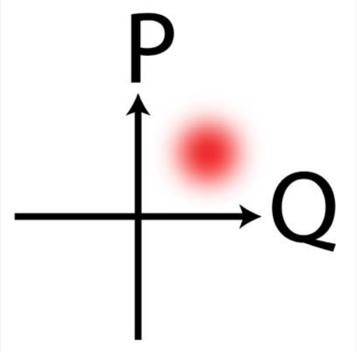
"quadratures": analogous to position and momentum

Classical



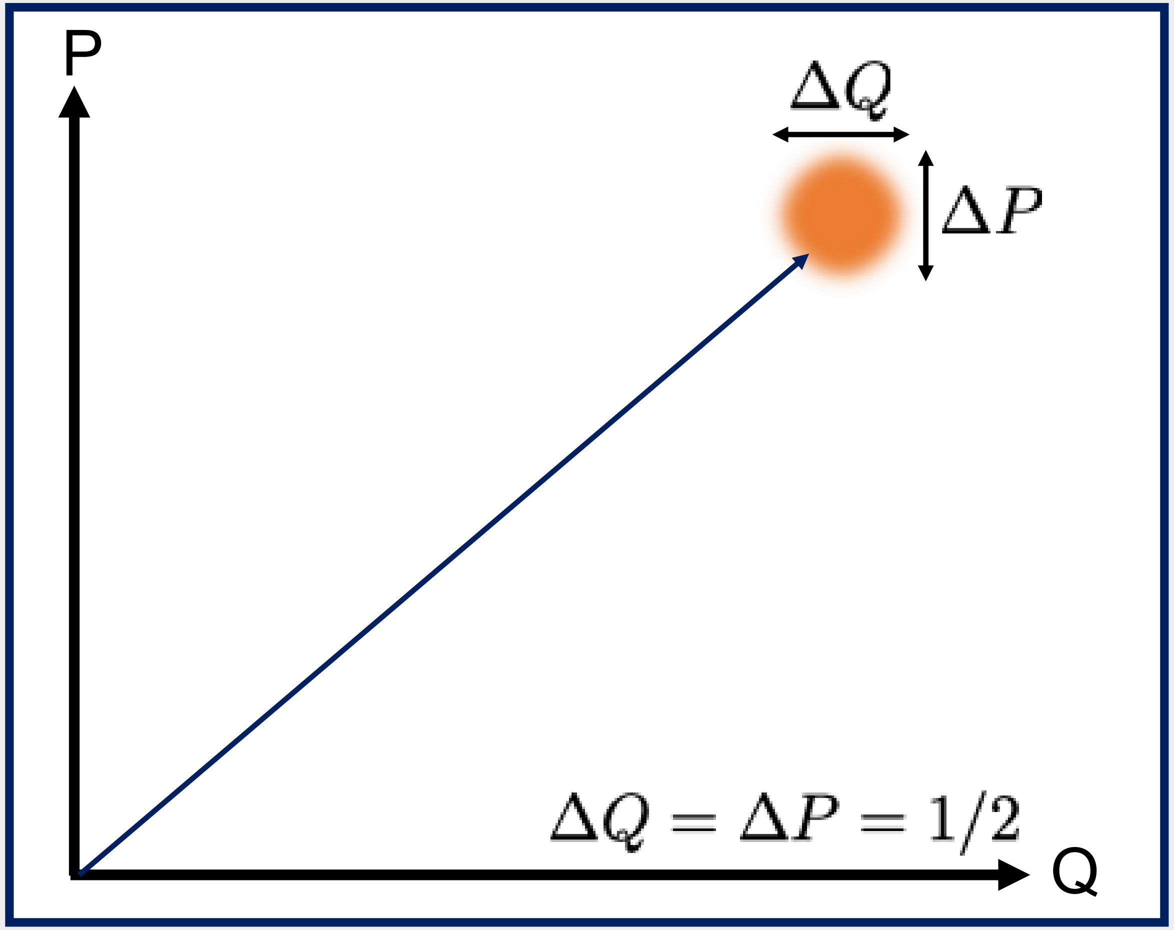
Q, P can be definite

Quantum



$$\Delta Q \Delta P \geq 1/4$$

Shot noise in phase space



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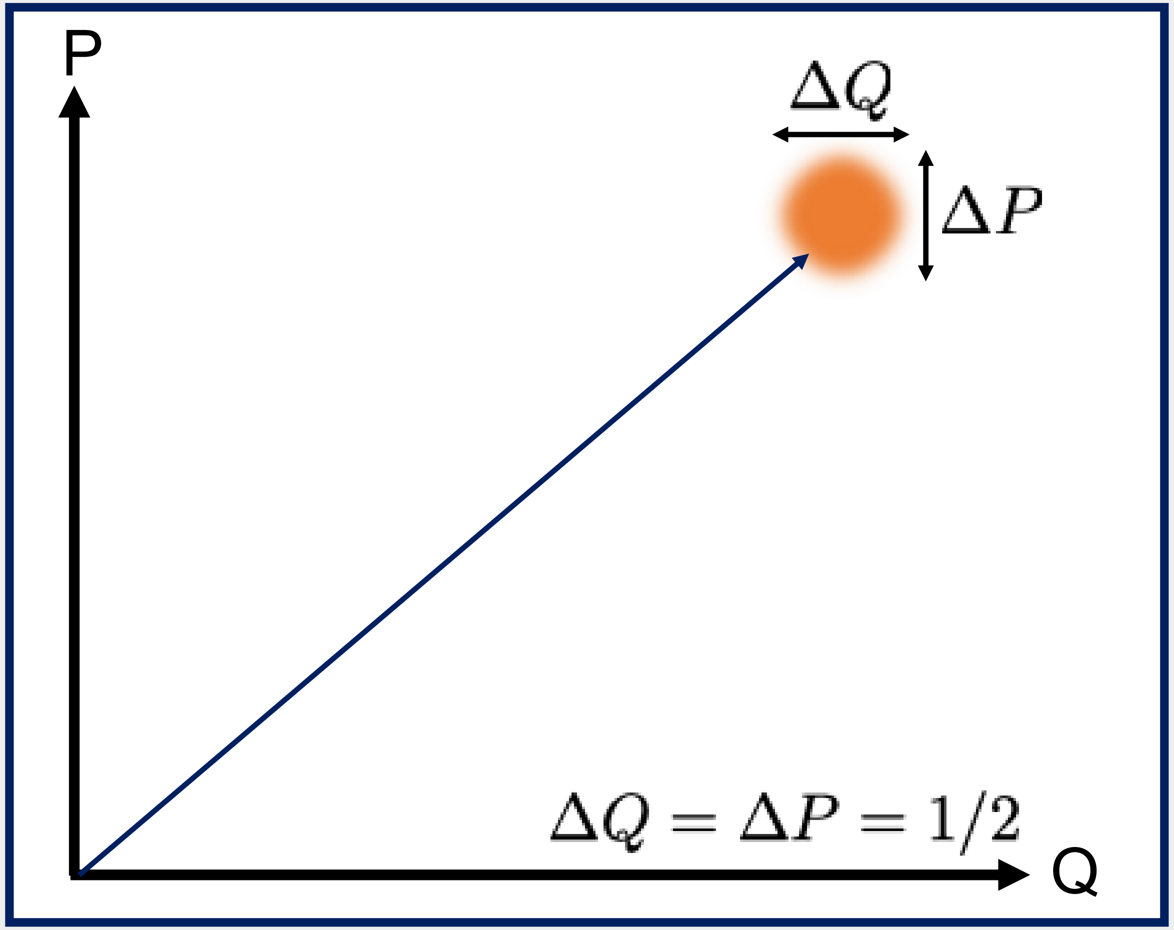

Energy (and thus photon number) given by: n = r^2 = $Q^2 + P^2$.

From the diagram:

$$\Delta n = \Delta r^2 = 2r\Delta r = \sqrt{n}$$

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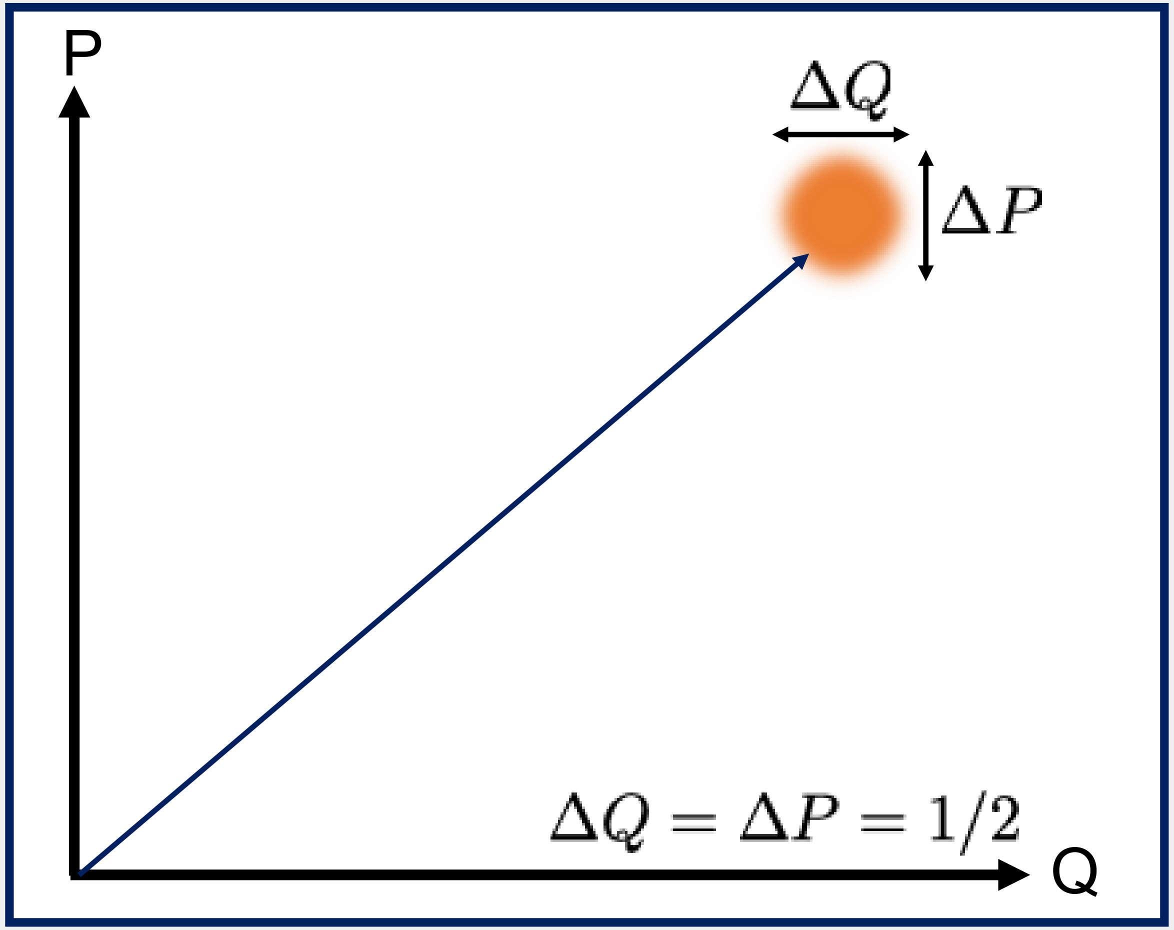
Shot noise in phase space



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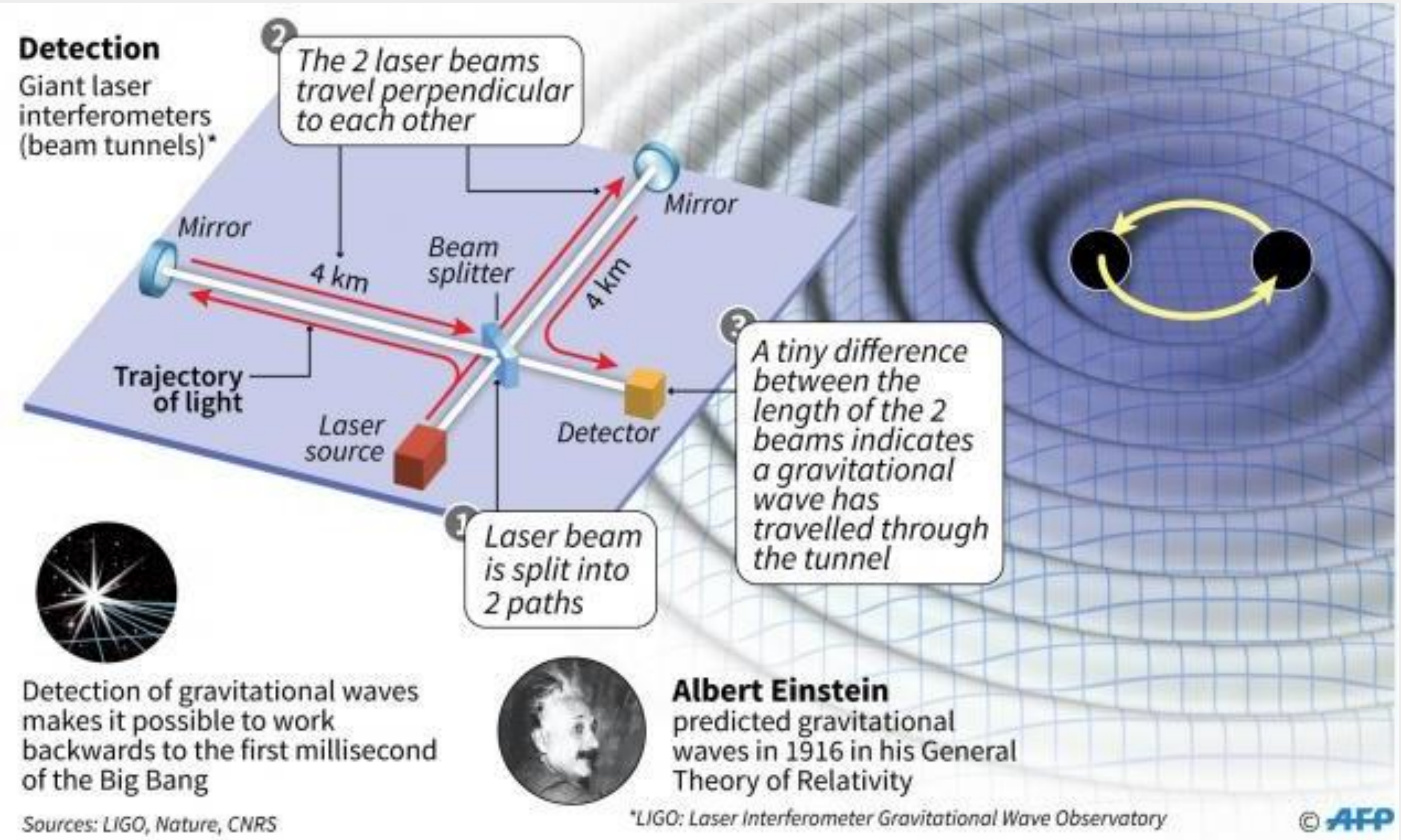
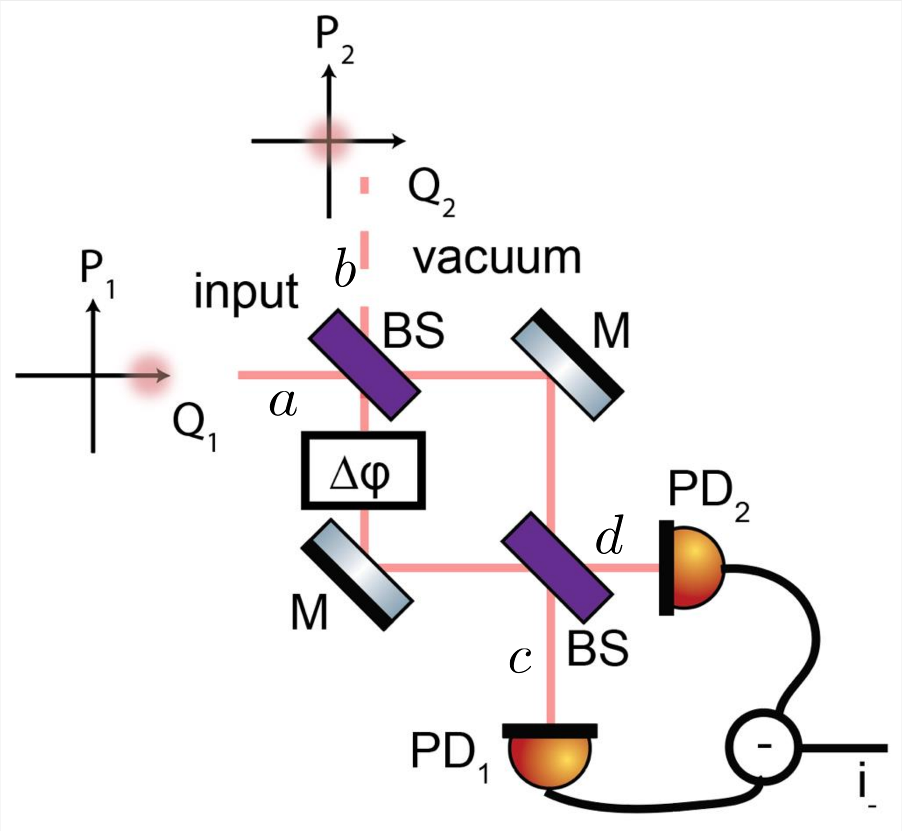
These intensity fluctuations are called **shot noise,** and result from *superposition of quantum noise with a classical deterministic field*.

Detecting gravitational waves with light



Shot noise limit in interferometry



$$i_- = c^\dagger c - d^\dagger d$$

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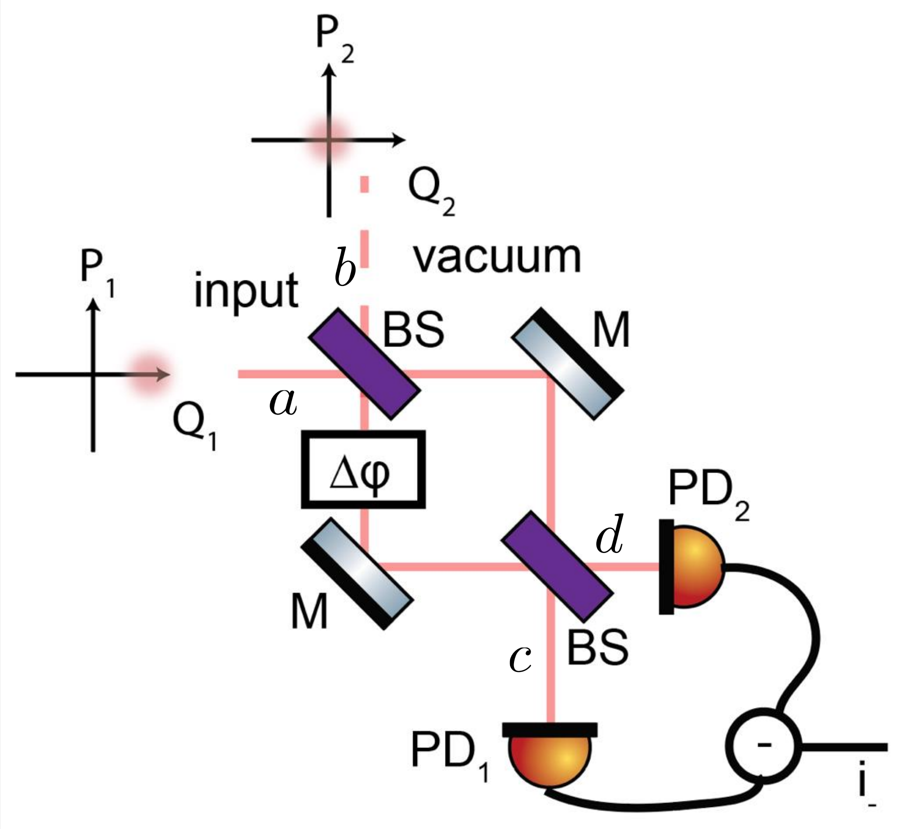

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$$(\Delta\phi)^2 \sim \frac{(\Delta Q_2)^2}{\langle n \rangle} \sim \frac{1}{\langle n \rangle}$$

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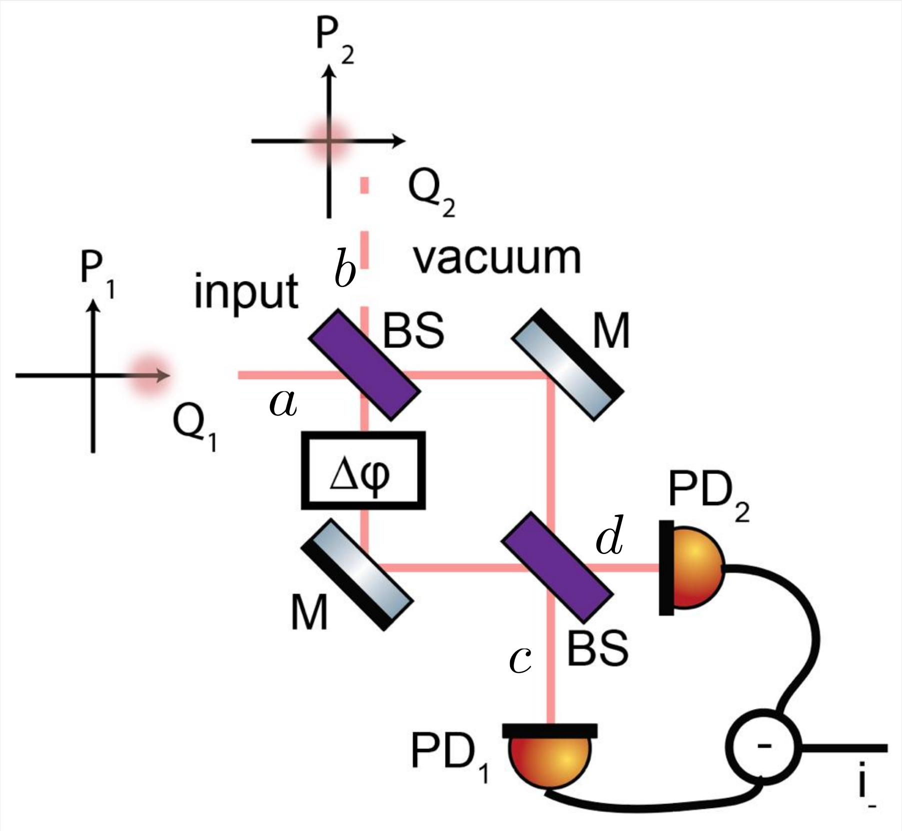

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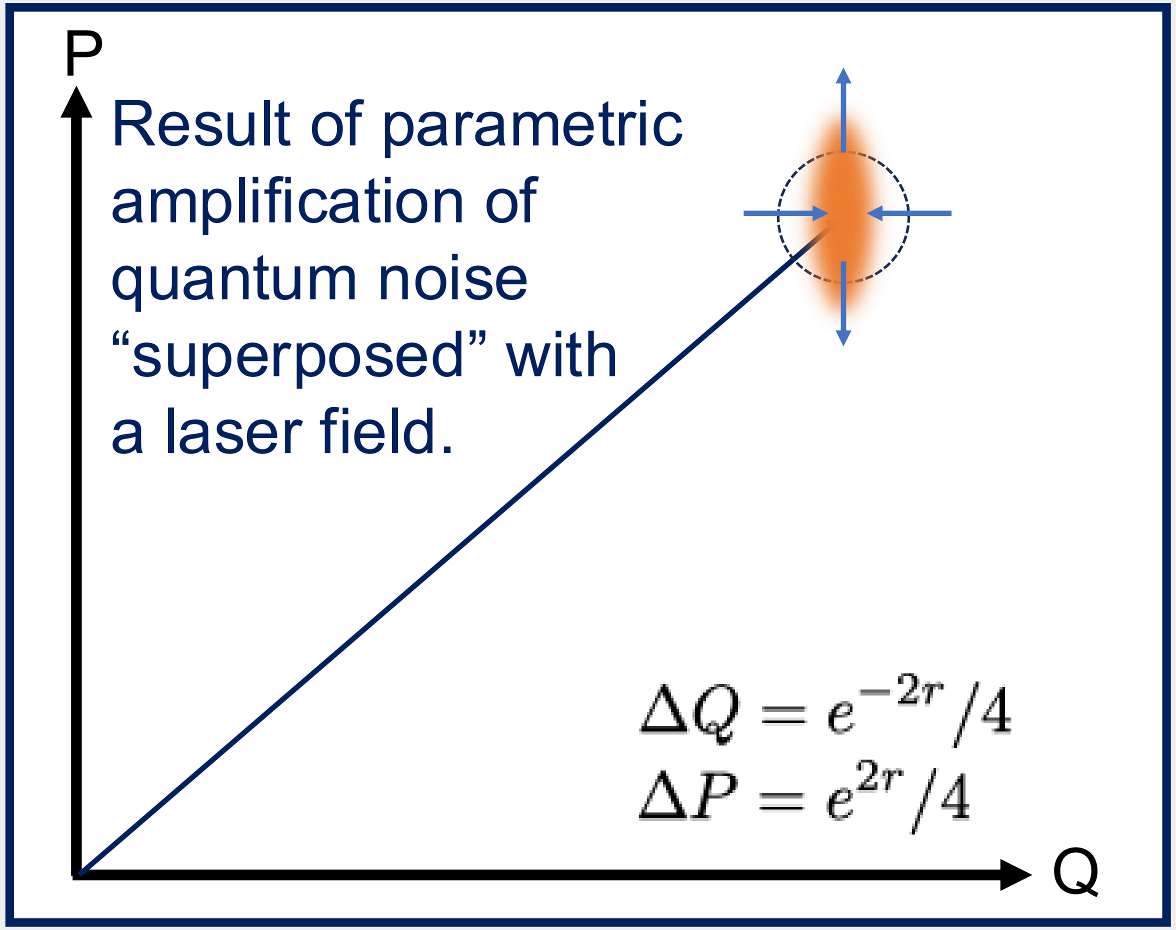
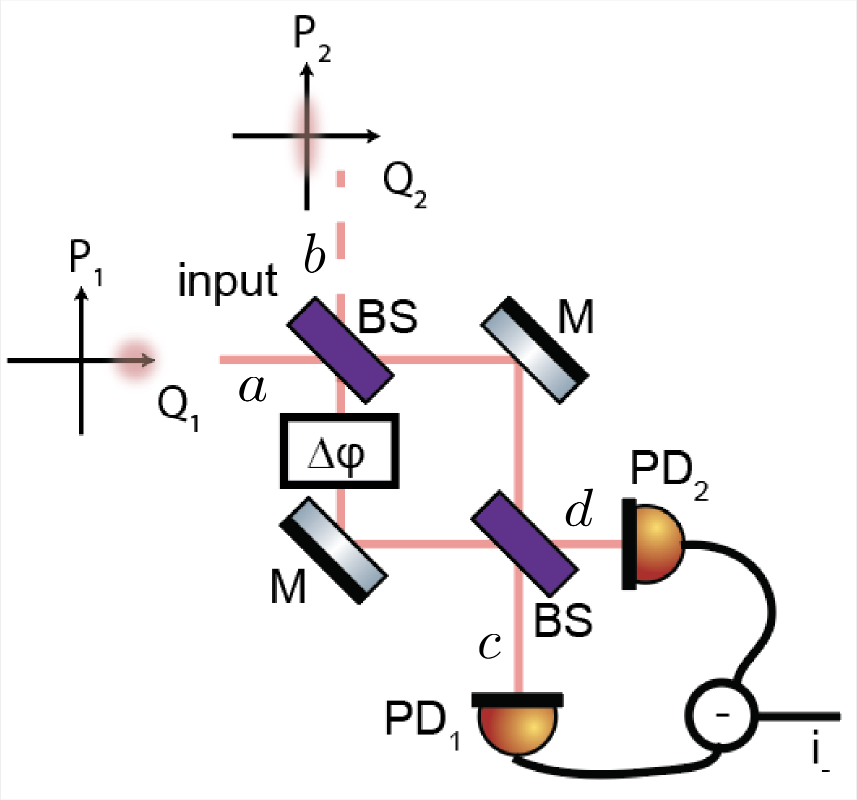
The minimum detectable phase is related to the inverse number of photons times the vacuum fluctuations in the second input (“b”) port! This is the **shot noise limit.**

Reducing quantum noise by squeezing



Quantum light interferometry



$$i_- = c^\dagger c - d^\dagger d$$

$$(\Delta\phi)^2 \sim \frac{(\Delta Q_2)^2}{\langle n \rangle} \sim \frac{e^{-2r}}{\langle n \rangle}$$

Sensitivity increases because the quantum noise in the "*b*" port has been compressed (core principle of the field of *quantum metrology*)*.

Enabled the Nobel Prize in Physics (2017) for gravitational wave discovery.

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*Want to learn more? Take AEP 4400 Nonlinear and Quantum Optics! ☺

Some closing thoughts