AEP 4400 / 5400 Nonlinear and Quantum Optics

Practice Final Exam

Your name: ______________________________

Question 1: ______________________________ /18

Question 2: ______________________________ /20

Question 3: ______________________________ /26

Question 4: ______________________________ /36

Total: ______________________________ /100

Instructions: 150 minutes is recommended for this practice exam. There are four questions. You may reference your notes, the course lecture notes, the textbook, and previous homework assignments. You may quote results derived in class, discussion, or problem sets without re-deriving them, unless explicitly asked. Note that for questions which require you to show a result: if the result is given, you may be able to do the subsequent parts of the problem even if you do not show the intended result.

I. SHORT QUESTIONS [18 POINTS]

(a) [5 points] Answer "true" or "false" and explain your answer: A collection of anharmonic oscillators with the potential $V(\mathbf{r}) = \frac{1}{2}m\omega_0^2\mathbf{r}^2 + \frac{1}{4}mb(\mathbf{r}\cdot\mathbf{r})^2$ can produce a $\chi^{(2)}$ response (i.e., a polarization quadratic in the field).

(b) [8 points] Consider a Lorentz oscillator whose potential is of the form

$$V(x,y,z) = \frac{1}{2}m\left[\omega_{\parallel}^2\begin{pmatrix}x & y\end{pmatrix}^T\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \omega_{\perp}^2 z^2\right]. \tag{1}$$

Find an expression for the susceptibility assuming a number density n of oscillators with charge q and mass m. Hint, in the principal axis basis of three directions $\hat{c}_1, \hat{c}_2, \hat{c}_3$, one may express the susceptibility via $\chi = \chi_1\hat{c}_1\hat{c}_1 + \chi_2\hat{c}_2\hat{c}_2 + \chi_3\hat{c}_3\hat{c}_3$ where $\hat{c}\hat{c}$ is a tensor that projects a vector onto the direction $\hat{c}$.

(c) [5 points] For a coherent state $|\alpha\rangle$ such that $a|\alpha\rangle = \alpha|\alpha\rangle$, find the expectation value $\langle\alpha|n^2|\alpha\rangle$, where $n = a^\dagger a$.

II. FREQUENCY-DOMAIN PICTURE OF HARMONIC GENERATION [20 POINTS]

Consider qth harmonic generation of a pulse propagating in a nonlinear crystal. Neglecting transverse dynamics, the equation of motion for the envelope field of the q harmonic, v_q is given by (in the lab frame)

$$\left(\partial_z + \frac{1}{v_q}\partial_t\right)A_q(z,t) = i\alpha(A_1(z,t))^q, \tag{2}$$

where α is a constant and A_1 is the field at the fundamental (first harmonic) which moves at velocity v_1.

(a) [6 points] Write down a formal integral solution for $A_q(z,\omega)$, where $A_q(z,\omega) = \int dt e^{i\omega t}A_q(z,t)$. You may assume that $A_q(0,t) = 0$ and that the pump is undepleted. Your answer should be of the form

$$A_q(z,\omega) = \int_0^z dz' \, [...]. \tag{3}$$

(b) [10 points] Consider the case in which $A_1(0,t)$ is a Gaussian pulse propagating at group velocity $v_1 \neq v_q$: $A_1(0,t) = A_0 e^{-t^2/\tau^2}$ with τ a measure of the pulse duration. Using this, find an expression for $A_q(z,\omega)$. There should be no integrals in your final answer.

(c) [4 points] Show that if $v_1 = v_q$, the resulting harmonic pulse has a pulse duration which is $\sqrt{q}$ times shorter than the fundamental.

III. QUANTUM LANGEVIN EQUATION FOR A LINEAR AMPLIFIER [26 POINTS]

In this problem, we will develop a description of gain in a linear amplifier. Classically, a linear amplification, in time-domain, is described by an equation

$$\dot{A} = gA, \tag{4}$$

where $g > 0$ and A is a proportional to the complex amplitude of the electric field.

(a) [4 points] Consider as a possible quantization $\dot{a} = ga$ where $[a(0), a^\dagger(0)] = 1$. Find $a(t)$ and evaluate $[a(t), a^\dagger(t')]$. Why is this quantization approach not valid?

(b) [6 points] To get to the correct quantization, consider a small time-increment Δt such that $g\Delta t \ll 1$. Find an operator $N(t)$ such that

$$a(t + \Delta t) = (1 + g\Delta t)a(t) + N(t) \tag{5}$$

preserves the commutation relations of $a, a^\dagger$. You should assume that $N(t)$ describes an auxiliary noise mode which commutes with a and $a^\dagger$.

(c) [6 points] Show that the relation, in the limit $\Delta t \to 0$ can be converted into a differential equation of the form $\dot{a} = ga + F(t)$ where $F(t)$ is a quantum Langevin force. Show that $[F(t), F^\dagger(t')] = \text{const.} \times \delta(t - t')$ as $\Delta t \to 0$ and find the value of the constant.

(d) [10 points] Find an expression for the quadrature variance of two orthogonal quadratures $(\Delta X_\theta)^2, (\Delta X_{\theta+\frac{\pi}{2}})^2$ at time t assuming that the initial state of both the light field and the noise modes (which generate the Langevin term) are in the vacuum state. Define the quadrature to be $X_\theta = \frac{1}{\sqrt{2}}(ae^{-i\theta} + a^\dagger e^{i\theta})$. You may find useful that the solution to (c) is given by

$$a(t) = a(0)e^{gt} + \int_0^t dt' e^{g(t-t')} F(t'). \tag{6}$$

IV. ELECTRON-PHOTON INTERACTION [36 POINTS]

We will consider a quantized model of a high-energy electron interacting with light in one dimension. You can imagine that the electron and photon are confined in the transverse direction and propagate freely along some direction x. When the electron is sufficiently fast-moving (for example, for electron beams in electron microscopes), the electron's energy-momentum dispersion can be approximated as *linear*:

$$E_k = \frac{\hbar^2 k^2}{2m} \approx \frac{\hbar^2 k_0^2}{2m} + \hbar v(k - k_0), \tag{7}$$

where $\hbar k_0$ is the average momentum of the electron wavepacket, and we have assumed that its momentum does not change too much relative to $\hbar k_0$. In the first part of the problem, we will work out the conditions under which the electron can efficiently interact with light.

(a) [4 points] Consider the process of spontaneous emission, in which an electron emits a photon. Write down energy and momentum conservation equations describing this process. You may take the initial electron wavevector to be k and the emitted photon wavevector to be q. Do not assume anything about the photon dispersion, and take its frequency to be ω_q.

(b) [4 points] Solve your equation in (a) to find a relationship between the photon phase velocity $v_q \equiv \omega_q/q$ and the electron velocity v.

(c) [2 points] The result above is typically referred to as a phase-matching condition. Why?

In subsequent parts of this problem, we will study a quantized model of electron-photon interactions. You do not need the results from (a-c) to solve the below parts. A model of a single-mode propagating light field interacting with a high-energy free electron, in one spatial dimension, is given by a unitary evolution of the form[1]

$$U = \exp\left[g\left(ab^\dagger - a^\dagger b\right)\right], \tag{8}$$

where a is the usual raising and lowering operator, and $b, b^\dagger$ is a so-called electron lowering/raising operator, defined such that

$$b|k\rangle = |k - q\rangle, b^\dagger|k\rangle = |k + q\rangle, \tag{9}$$

where k is the wavevector of the electron plane wave and q is the photon wavevector. You may assume $k \gg q$.

(d) [3 points] Show that $[b, b^\dagger] = 0$.

(e) [6 points] Evaluate $a' = U^\dagger a U$. You may find useful the Campbell formula

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \dots . \tag{10}$$

(f) [4 points] Suppose that the initial state of the system is $|k, 0\rangle$, corresponding to the electron having wavevector k (and momentum $\hbar k$) and the light

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[1] Strictly speaking, this is in the interaction picture, but you should not worry too much about this here.

field being in the vacuum state. Find the expectation values of $a, a^\dagger$ and $a^\dagger a$.

(g) [8 points] Suppose that the light field is in the state $|0\rangle$ and the electron is in the state

$$|\psi\rangle = \int \frac{dk}{2\pi}\, c(k)|k\rangle, \tag{11}$$

corresponding to a wavepacket whose spatial profile is set by the Fourier superposition coefficients [2]. Show that for generic $c(k)$

$$\langle a\rangle = \beta, \tag{12}$$

and find an expression for β. In doing this problem, you may take the state to be normalized so that $\int \frac{dk}{2\pi}|c(k)|^2 = 1$ and $\langle k|k'\rangle = 2\pi\delta(k-k')$.

(h) [5 points] Consider the case in which the wavenumber of the light field q is much less than the spread Δk of the wavepacket. You may take $q \ll \Delta k \ll k_0$. Show that $\langle b^m\rangle \approx \langle b\rangle \approx 1$ in the stated limit (you may assume $mq \ll \Delta k$), and from this, argue that the transformation in (e) can be approximated as a displacement transformation on the field.

This shows you the so-called classical limit of electron-photon interaction. When the electron wavepacket is sufficiently broad in momentum-space such that recoil does not change the electron wavefunction much, the electron may be treated as a classical current.

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[2] In particular, the electron wavefunction corresponding to this state is $\psi(x) = \langle x|\psi\rangle \sim \int \frac{dk}{2\pi}\, e^{ikx} c(k)$.