AEP 4400 / 5400 Nonlinear and Quantum Optics

Practice Exam 2

Your name: ______________________________

Question 1: ______________________________ /21

Question 2: ______________________________ /19

Question 3: ______________________________ /30

Question 4: ______________________________ /30

Total: ______________________________ /100

Instructions: You have 90 minutes to complete this exam. Do not start until instructed to do so. There are four questions. You may reference your notes, the course lecture notes, the textbook, and previous homework assignments. You may quote results derived in class, discussion, or problem sets without re-deriving them, unless explicitly asked. Note that for questions which require you to show a result: if the result is given, you may be able to do the subsequent parts of the problem even if you do not show the intended result. **Please turn in the exam sheets, your blue book, and any scrap paper you used.**

I. SHORT QUESTIONS [21 POINTS]

For the first two questions, answer true or false, and explain your answer.

(a) [6 points] Consider the nonlinear beam propagation equation

$$\partial_z A(\boldsymbol{\rho}, z) = i\kappa |A(\boldsymbol{\rho}, z)|^2 A(\boldsymbol{\rho}, z). \tag{1}$$

In this equation an initial Gaussian solution will evolve into a narrower Gaussian beam.

(b) [6 points] Consider a bulk linear medium with index of refraction $n(\omega)$. You are told that $\frac{d(n\omega)}{d\omega} < 0$ for all frequencies within a bandwidth of interest. Now consider a pulse whose spectrum is confined to this bandwidth. In the far-field, the pulse will necessarily develop a chirp such that shorter wavelengths are on the leading edge of the pulse.

(c) [9 points] Consider the process of four-wave mixing in which two photons of a pump wave at frequency ω and wavenumber k split up into one photon at frequency $\omega + \Delta, k + q$ and $\omega - \Delta, k - q$. Assuming the fields are all collinear and all evolution is along the propagation direction z, write three valid Manley-Rowe relations describing the evolution of these fields.

II. FOCUSING A GAUSSIAN BEAM [19 POINTS]

Consider a well collimated Gaussian beam with envelope $A(\boldsymbol{\rho}, z = 0) = A_0 e^{-\rho^2/w^2}$. At $z = 0$, the wave encounters a thin lens. The lens can be thought of as a shaped piece of transparent material with linear index n_0 (nonlinearity can be neglected) surrounded by vacuum, and a thickness that depends on the distance from the optical axis $\rho = 0$. The thickness of the lens can be taken to be

$$t(x, y) = t_0 - \alpha\rho^2. \tag{2}$$

You may assume that $z \ll b$ where b is the confocal parameter of the Gaussian beam. The lens is thin enough to only act by applying a phase onto the beam.

(a) [6 points] The wave at the exit of the lens will have the form $A(\boldsymbol{\rho}, z) = A_0 e^{i\varphi(\rho)}$. Find $\varphi(\rho)$. This phase is defined as the phase picked up by each radial position ρ of the beam after traversing t_0 distance (i.e., between the entrance and exit planes of the lens). Hint: at a radial position ρ, $t(\rho)$ distance is associated with index n_0 and $t - t(\rho)$ distance is associated with index 1.

(b) [5 points] By examining the instantaneous wavevector or otherwise, find the sign of α that will lead to the beam converging towards the optical axis (focusing) for $z > 0$.

(c) [8 points] This beam will come to a focus at some length f (the focal length). What is that length? You may find the following integral useful (although there is a simple geometric argument that yields the correct answer):

$$\int_{-\infty}^{\infty} dx\, e^{-ikx-ax^2} = \sqrt{\frac{\pi}{a}} e^{-\frac{k^2}{4a}}. \tag{3}$$

If you go the route of evolving the beam up to length z, you should keep in mind that the initial beam being large should be taken to mean that $k_0\alpha \gg w^{-2}$.

III. SATURABLE ABSORBER NONLINEARITY [30 POINTS]

Consider a pulse propagating in a single-mode waveguide which is dispersionless. Additionally, it has no direct Kerr nonlinearity, but it does have a nonlinear absorption. The nonlinear absorption in question is called saturable absorption, and the amplitude absorption coefficient $\alpha = \alpha(I) = \frac{\alpha}{1+I/I_s}$ where I is the optical intensity and I_s is called the saturation intensity. The resulting pulse propagation equation in this situation is given by

$$\partial_z A(z, t') = -\frac{\alpha}{1 + I(z, t')/I_s} A(z, t'). \tag{4}$$

The equation is written in units such that $I(z, t') = |A(z, t')|^2$.

(a) [6 points] Find an equation of motion for the intensity $I(z, t')$ of the form $\partial_z I(z, t') = [...]$.

(b) [9 points] By integrating your equation, find $I(z)$. You should get a transcendental equation. Hint: you should get that $I = I(z)$ satisfies: $Ie^{I/I_s} = f(I_0, z)$ for some function of $I_0 = I(0)$ and z.

(c) [10 points] How does $I(z)$ behave for $I(0, t') \ll I_s$ and $I(0, t') \gg I_s$? Use this to sketch how the pulse might look after passing through a saturable absorber.

(d) [5 points] Explain how a saturable absorber can be used to make a pulse shorter.

IV. MODULATION INSTABILITY OF PULSES [30 POINTS]

Consider a single-mode optical fiber with only second-order dispersion. The envelope field $A(z, t')$ in the co-moving frame satisfies

$$\partial_z A(z, t') = -\frac{i\beta_2}{2}\partial_{t'}^2 A(z, t') + i\gamma|A(z, t')|^2 A(z, t'). \tag{5}$$

We would like to see if a constant intensity solution is stable to nonlinear propagation. Consider a solution of the form

$$A(z, t') = A_0(z) + \delta A(z, t'), \tag{6}$$

where $\delta A(z, t')$ is a weak modulation satisfying $|\delta A(z, t')| \ll |A_0|$ and A_0 is a strong pump plane wave.

(a) [10 points] Derive a coupled set of linear differential equations for the perturbation $\delta\tilde{A}(z, t'), \delta\tilde{A}^*(z, t')$ where $A(z, t') = \delta\tilde{A}(z, t')e^{i\Omega_{NL}z}$. Ω_{NL} should be such that your linear equation has only constant coefficients.

(b) [5 points] By going in Fourier space, defining $A(z, t) = \int \frac{d\Omega}{2\pi} e^{-i\Omega t} A(z, \Omega)$, write coupled differential equations of the form

$$\partial_z \begin{pmatrix} \delta\tilde{A}(z,\Omega) \\ (\delta\tilde{A}^*)(z,\Omega) \end{pmatrix} = M(\Omega) \begin{pmatrix} \delta\tilde{A}(z,\Omega) \\ (\delta\tilde{A}^*)(z,\Omega) \end{pmatrix}, \tag{7}$$

where $M(\Omega)$ is a matrix whose entries you must specify [1].

(c) [10 points] Find the eigenvalues $\lambda(\Omega)$ of M and describe the conditions required for fluctuations at some frequency Ω to be amplified. Consider this for all possible signs of β_2 and γ.

(d) [5 points] It is sometime said that anomalous dispersion nonlinear effects lead to greater noise amplification than normal dispersion nonlinear effects. Explain in light of your analysis why that makes sense. Assume $\gamma > 0$.

[1] $(\delta\tilde{A}^*)(z,\Omega)$ is the Fourier transform of $\delta\tilde{A}^*(z,t)$ and is equal to $((\delta\tilde{A})(z,-\Omega))^*$ though you will not need this identity.