

AEP 4230 / PHYS 4230 Statistical Thermodynamics

Practice for Final Exam

Your name: _____

Question 1: _____/30

Question 2: _____/20

Question 3: _____/30

Question 4: _____/20

Total: _____/100

During the final exam, you will have 120 minutes to complete an exam of a similar format and difficulty to this practice exam.

I. GRAPHENE (30 POINTS)

Graphene is a single sheet of carbon atoms bonded into a two-dimensional hexagonal lattice. The band structure of electrons in graphene is such that the single-particle excitations behave as relativistic Dirac fermions with a spectrum at low energies that can be approximated by $E_\sigma(\mathbf{k}) = \sigma\hbar vk$ where $\sigma = \pm 1$ and $v \approx 10^6 m/s$. There is on top of this, a spin degeneracy of $g = 2$ since the single-particle excitations are electrons and holes.

(a) [12 points] If at zero temperature all negative energy states are occupied and all positive energy states are empty, show that the chemical potential $\mu(T)$ is zero at all temperatures. Hint: As the temperature increases from zero, the density of electrons should be constant. Therefore, the chemical potential should impose a constraint that the total number of particles, as prescribed by the Fermi-Dirac distribution, is constant.

(b) [10 points] Show that the mean excitation energy of this system at finite temperature satisfies

$$E(T) - E(0) = 4A \int \frac{d^2k}{(2\pi)^2} \frac{E_+(\mathbf{k})}{e^{\beta E_+(\mathbf{k})} + 1}. \quad (1)$$

(c) [8 points] Find an expression for the specific heat at low temperatures (at high temperatures, the linear dispersion no longer describes electronic excitations in graphene). Express your result in terms of the appropriate $f_m^-(z)$.

II. AVERAGING FILTER (20 POINTS)

An averaging filter is one that averages its input power over some fixed time interval Δt , taking a signal $y(t)$ and returning a signal $w(t)$ such that

$$w(t) = \frac{1}{\Delta t} \int_t^{t+\Delta t} dt' y(t'). \quad (2)$$

(a) [10 points] Find the transmission function of this filter $t(\omega)$ defined such that $w(\omega) = t(\omega)y(\omega)$. Sketch the (modulus) square of the transmission $|t(\omega)|^2$.

(b) [10 points] Suppose that y can be approximated as white noise for frequencies

$\omega \ll (\Delta t)^{-1}$. Find the variance in the filtered output w is given by in terms of δt and $S_y(0)$, the *two-sided* power spectral density of y at zero *angular* frequency.

III. THREE SHORT QUESTIONS (30 POINTS)

(a) [10 points] Describe a technique to use a barometer (a device that measures pressure) to measure height. In doing so, estimate the pressure resolution of the barometer (in atmospheres) needed to measure a height of 100 feet. You may find useful the following constants $k_B = 1.38 \times 10^{-23} J/K$, $g \approx 10 m/s^2$, $\bar{m} = 29 \times 1.67 \times 10^{-27} kg$. The quantity \bar{m} is the average molecular mass of molecules in the atmosphere. You do not need to calculate this number too precisely: the nearest order of magnitude is OK.

(b) [10 points] Find an expression for the energy density per unit frequency $u(\omega) = \frac{1}{V} \frac{dU}{d\omega}$ of photons in thermal equilibrium in free space. Find the resulting expression for the energy density if one takes a classical limit $\hbar\omega \ll k_B T$ of the energy density. This is called the Rayleigh-Jeans law. Show that if one makes the approximation $\hbar\omega \ll k_B T$ for all frequencies (which of course, does not make sense), the energy diverges. This is called an ultraviolet catastrophe, and motivated the development of quantum mechanics.

(c) [10 points] Estimate the ratio between (a) the thermal deBroglie wavelength of a neutron at room temperature (the mass of a neutron is 1.66×10^{-27} kg) and (b) the minimum wavelength of a phonon in a three-dimensional solid as computed from the Debye model. You may take the density of a solid supporting phonons to be $10^{29} m^{-3}$.

IV. FREE-ENERGY PERTURBATION THEORY AND SPECIFIC HEAT OF AN ANHARMONIC OSCILLATOR (20 POINTS)

Suppose we have a classical system with Hamiltonian H_0 and corresponding partition function Z . Suppose now we have the Hamiltonian $H_0 + V$ where V is

a perturbation (defined such that $V \ll H_0$ in the relevant way such that you can expand quantities in V).

(a) [10 points] Find the free energy up to second order in V by expanding the partition function in V . Denote expectation values taken with respect to the unperturbed probability distribution as $\langle A \rangle_0$.

(b) [10 points] Using this, find the lowest-order non-zero correction to the specific heat for a classical anharmonic oscillator with Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 x^2 + ax^4$. You may assume that a is sufficiently small.