Unit 4: Quantum mechanical description of the electromagnetic field in matter

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Contents

We have thus far spent our time in this course studying how the propagation of light is modified by a material. In the linear regime, in which the material polarization is linear in the total electric field, the overall effect is to establish a refractive index for light. In the nonlinear regime in which the material polarization is nonlinear, a variety of new effects emerge such as the generation of new frequencies, the electro-optic effect, and the intensity-dependent refractive index. In all of our treatment thus far, our treatment of the electromagnetic field and matter has been according to classical physics, although we have explained a few things in terms of a heuristic photon picture.

At the same time, the fundamental description of light and its interaction with matter is based on quantum mechanics. In standard treatments of quantum mechanics, it is typical to quantize matter: to quantize electrons, nuclei, and other matter particles and study their quantum properties based on the dynamics of the corresponding wavefunction. Based on the time-development of the wavefunction, expectation values of Hermitian operators (observables) can be computed. It is in some sense only expectation values of Hermitian operators that we can physically observe. Although it is not typical in standard courses of quantum mechanics to quantize the light field, it can be done and conceptually the quantum mechanics of the electromagnetic field is similar to the quantum mechanics of matter. Compared to electromagnetic field quantization as is done for example in quantum field theory, our quantization has to be able to describe the quantum properties of light propagating in matter, which requires accounting for the effect of polarization in a quantum mechanically consistent way.

In this second part of the course (Units 4 and 5), we will treat the electromagnetic field quantum mechanically. Such a treatment is necessary to describe the statistical properties of light, which are important for describing spontaneous emission, noise in light sources, fundamental limitations to optical amplifiers, and nonclassical correlations such as entanglement and squeezing which are of importance for quantum-enhanced sensing.

In Unit 4, we start by quantizing the electromagnetic field in matter. The outcome of this procedure is a Hamiltonian for the electromagnetic field expressed in terms of electromagnetic field *operators*: specifically the vector potential and the displacement field, which will have canonical commutation relations much like the position and momentum of a particle. Using this Hamiltonian, we will explore the stationary states (eigenstates) and important superpositions of them which correspond to commonly realized quantum states of light. From there, we will describe how we measure the quantum properties of light, and in doing so, develop a theory of photodetection. With the apparatus of photodetection theory, we can then describe different types of correlation measurements of light and under what conditions a certain measurement cannot be realized classically [1].

In many cases, light states experience attenuation or amplification which correspond to *non-conservative* or *open system* dynamics classically. A quantum mechanically consistent treatment of such open system dynamics requires the introduction of *quantum noise sources* whose effect is typically to degrade quantum correlations and degrade signal-to-noise ratios. Such effects lead to important limits on devices such as optical amplifiers, with important implications for optical communications.

In Unit 5, on the quantum optical description of nonlinear optical phenomena, we will use the apparatus developed in this unit describe how nonlinear media lead to the generation of quantum mechanical states of light such as entangled photons and squeezed light. And we will conclude by illustrating how quantum light states allow measurement sensitivities which can exceed what is possible classically.

I. QUANTIZATION OF THE ELECTROMAGNETIC FIELD IN MATTER

A. Electromagnetic Lagrangian of a lossless nonlinear medium

We start by developing a quantum description of the electromagnetic field in a material. Such a description is called *macroscopic quantum electrodynamics.* We follow a standard procedure in quantum field theory: (a) identify a Lagrangian whose Euler-Lagrange equations reproduce the classical equations of motion, (b) identify the corresponding Hamiltonian via the Legendre transform, and (c) canonically quantize

[1] This chapter was written with helpful contributions from Jamison Sloan.

the fields by introducing canonical commutation relations for the complementary dynamical variables. These canonical commutation relations must reproduce the classical description in the appropriate classical limit. This procedure is called canonical quantization. We will assume familiarity with Lagrangian and Hamiltonian mechanics, as well as Maxwell's equations based on the vector and scalar potentials. These main results are reviewed in Appendices 1 and 2, and the reader is encouraged to look there for more details.

In Unit 1, we discussed the energy of the electromagnetic field in a nonlinear medium and in so doing, introduced a Lagrangian for the electromagnetic field in matter which reproduced the Maxwell equations in the presence of polarization. We found that the Lagrangian density was given in terms of the electric and magnetic fields via

$$\mathcal{L} = \frac{\epsilon_0}{2}(\mathbf{E}^2 - c^2\mathbf{B})^2 + \mathcal{L}_{\text{pol}}, \tag{I.1}$$

where

$$\mathcal{L}_{\text{pol}} = \epsilon_0 \sum_{n=1}^{\infty} \frac{1}{n+1} \chi^{(n)}_{i_0 i_1 i_2 \cdots i_n} E_{i_0} E_{i_1} \cdots E_{i_n} \equiv \epsilon_0 \sum_{n=1}^{\infty} \frac{1}{n+1} \chi^{(n)} : \mathbf{E}^{\otimes(n+1)}. \tag{I.2}$$

Let us now find the corresponding Euler-Lagrange equations for this Lagrangian. Before we do this, we will do a change of variables, from $\mathbf{E}$ and $\mathbf{B}$ to $\mathbf{A}$ and ϕ, the vector and scalar potentials. As a reminder, the fields are related to the potentials by $\mathbf{E} = -\partial_t \mathbf{A} - \nabla\phi$ and $\mathbf{B} = \nabla \times \mathbf{A}$. Importantly, we remember from electrodynamics that there is *gauge freedom*: we are allowed to change the potentials by a gauge transformation without changing the physical electric and magnetic fields. Recall that if we take $\phi \to \phi + \partial_t \chi$ and $\mathbf{A} \to \mathbf{A} - \nabla\chi$ for any function χ, we get the same fields. Therefore, when we work with potentials, it is useful to specify a gauge. A useful gauge is the so-called *Coulomb gauge* in which $\nabla \cdot \mathbf{A} = 0$[2].

Quantization is usually approached from the standpoint of the potentials because there are certain constraints which emerge naturally in the language of potentials that are important to track. We will see this shortly. The Euler-Lagrange equations

[2] The reader should convince themselves that if they are in a gauge in which $\nabla \cdot \mathbf{A} \neq 0$ then they can find a χ which makes the vector potential divergenceless and that this χ satisfies a Poisson equation.

for the scalar potential are[3]:

$$\frac{\partial \mathcal{L}}{\partial \phi} + \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} + \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j \phi)} = 0, \tag{I.3}$$

where the j denotes spatial indices which are summed over. The first and second terms are zero, as the only way ϕ appears in the equation is through its gradient. The relevant derivative is most easily evaluated by

$$\frac{\partial \mathcal{L}}{\partial(\partial_j \phi)} = \frac{\partial \mathcal{L}}{\partial E_i} \frac{\partial E_i}{\partial(\partial_j \phi)} = -\delta_{ij} \frac{\partial \mathcal{L}}{\partial E_i}. \tag{I.4}$$

The derivative with respect to the electric field is evaluated as

$$\frac{\partial \mathcal{L}}{\partial E_i} = \left(\epsilon_0 E_i + \epsilon_0 \sum_{n=1}^{\infty} \chi^{(n)}_{ii_1 \cdots i_n} E_{i_1} \cdots E_{i_n} \right) = D_i, \tag{I.5}$$

where $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$. In evaluating the derivative, I made use of the Kleinman symmetry of a lossless medium which allows us to freely permute the indices of $\chi^{(n)}$. This renders all $n+1$ terms generated in the product rule equal to each other. Combining these derivatives with the Euler-Lagrange equation gives

$$\partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j \phi)} = -\epsilon_0 \partial_j D_j = 0, \tag{I.6}$$

In other words, we get Gauss' law:

$$\nabla \cdot \mathbf{D} = 0. \tag{I.7}$$

This equation tells us that ϕ is not freely determined. Why? Because we can write

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon_0(-\partial_t \mathbf{A} - \nabla \phi) + \mathbf{P}). \tag{I.8}$$

In the Coulomb gauge, we may write

$$\nabla \cdot \mathbf{D} = -\epsilon_0 \nabla^2 \phi - \rho_b = 0 \implies \nabla^2 \phi = -\frac{\rho_b}{\epsilon_0}. \tag{I.9}$$

where $\rho_b = \nabla \cdot \mathbf{P}$ is the bound polarization charge. Therefore the potential is given in terms of the electric field itself (which determines uniquely the polarization) by Coulomb's law (this is why the gauge is called Coulomb gauge).

[3] As described in the relevant Appendix, we treat the field, its time derivatives, and its space derivatives all as independent variables.

So much for the scalar potential. Let us now look at the equation for the vector potential. For each component, A_i, the Euler-Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial A_i} + \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t A_i)} + \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j A_i)} = 0. \tag{I.10}$$

The first term vanishes while the other two are non-zero. The first of the two is

$$\frac{\partial \mathcal{L}}{\partial(\partial_t A_i)} = \frac{\partial \mathcal{L}}{\partial E_j} \frac{\partial E_j}{\partial(\partial_t A_i)} = -\delta_{ij} \frac{\partial \mathcal{L}}{\partial E_j} = -D_i. \tag{I.11}$$

The second of the two is evaluated using a similar chain-rule approach as:

$$\frac{\partial \mathcal{L}}{\partial(\partial_j A_i)} = \frac{\partial \mathcal{L}}{\partial B_k} \frac{\partial B_k}{\partial_j A_i} = -\frac{B_k}{\mu_0} \frac{\partial B_k}{\partial_j A_i}. \tag{I.12}$$

The derivative of the magnetic field is evaluated as

$$\frac{\partial B_k}{\partial_j A_i} = \frac{\partial}{\partial_j A_i} \epsilon_{krs} \partial_r A_s = \epsilon_{krs} \delta_{jr} \delta_{is} = \epsilon_{kji}, \tag{I.13}$$

where ϵ_{ijk} is the usual Levi-Civita symbol. Using this, we find

$$\frac{\partial \mathcal{L}}{\partial(\partial_j A_i)} = -\frac{B_k}{\mu_0} \epsilon_{kji}, \tag{I.14}$$

and so the Euler-Lagrange equation for the vector potential may be written as

$$-\partial_t D_i - \frac{1}{\mu_0} \epsilon_{kji} \partial_j B_k = 0 \implies \nabla \times \mathbf{H} = \partial_t \mathbf{D}, \tag{I.15}$$

where $\mathbf{H} = \mathbf{B}/\mu_0$. This is just the Ampere-Maxwell law in a medium. We may use these results to get an overall wave equation as follows. From the Ampere-Maxwell law and the definition of the potentials, we have

$$\nabla \times \nabla \times \mathbf{A} = \mu_0 \partial_t (\epsilon_0 \mathbf{E} + \mathbf{P}) \implies -\nabla \times \nabla \times \partial_t \mathbf{A} = -\frac{1}{c^2} \partial_t^2 \mathbf{E} - \mu_0 \partial_t^2 \mathbf{P}. \tag{I.16}$$

Using $-\partial_t \mathbf{A} = \mathbf{E} + \nabla \phi$ and $\nabla \times \nabla \phi = 0$, we have

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2} \partial_t^2 \mathbf{E} = -\mu_0 \partial_t^2 \mathbf{P}, \tag{I.17}$$

which is the Maxwell wave equation. From these, we can in principle back-track and get the magnetic Gauss law and the Faraday laws as well.

We have now shown that our postulated Lagrangian reproduces the classical Maxwell equations. Before quantizing the equations, we now develop the expression for the Hamiltonian, which is the classical energy of the system. You also know however that the Hamiltonian is the generator of time-evolution in quantum mechanics, and so the Hamiltonian will be a central object for us in our quantization.

B. Hamiltonian of a lossless nonlinear medium

The Hamiltonian H is typically expressed as an integral over the Hamiltonian density $\mathcal{H}$ (energy density) via

$$H = \int d^3r\ \mathcal{H}. \tag{I.18}$$

The Hamiltonian density follows from a Legendre transform of the Lagrangian. You may recall for a single particle, $H(p,q) = p\dot{q} - L(q,\dot{q})$ where $p = \frac{\partial L}{\partial \dot{q}}$ is the canonical momentum corresponding to the generalized coordinate q. The idea in field theory is the same where in our case the generalized coordinates are taken to be the potentials $\phi, \mathbf{A}$. The Hamiltonian density corresponding to our electromagnetic Lagrangian is:

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)}(\partial_t \phi) + \frac{\partial \mathcal{L}}{\partial(\partial_t A_i)}(\partial_t A_i) - \mathcal{L}. \tag{I.19}$$

The canonical momentum for the scalar potential

$$\Pi_0 = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = 0, \tag{I.20}$$

which is a manifestation of the fact that ϕ is not an independent degree of freedom. The canonical momentum density for the vector potential however is nontrivial and given by

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial(\partial_t A_i)} = -D_i. \tag{I.21}$$

Very importantly, the canonical momentum is the displacement field. The Hamiltonian density may now be written as

$$\mathcal{H} = -D_i(-E_i - \partial_i \phi) - \mathcal{L} = \mathcal{H}' + \mathbf{D} \cdot \nabla \phi, \tag{I.22}$$

where $\mathcal{H}' = \mathbf{D} \cdot \mathbf{E} - \mathcal{L}$. I have roped off the last term off because it does not contribute to the Hamiltonian. That is because

$$\int d^3r\ \mathbf{D} \cdot \nabla \phi = -\int d^3r\ \phi(\nabla \cdot \mathbf{D}) = 0. \tag{I.23}$$

Therefore, we will not keep track of it, and only consider

$$\mathcal{H}' = \mathbf{D} \cdot \mathbf{E} - \mathcal{L} = \epsilon_0 \sum_{n=1}^{\infty} \frac{n}{n+1} \epsilon^{(n)} : \mathbf{E}^{\otimes(n+1)} + \frac{\mathbf{B}^2}{2\mu_0}, \tag{I.24}$$

where

$$\epsilon^{(n)} = \chi^{(n)} \text{ if } n \geq 2; \text{ else } \epsilon^{(1)} = 1 + \chi^{(1)}, \tag{I.25}$$

where "1" represents the two-by-two identity matrix.

To quantize the fields, we follow the standard procedure in quantum mechanics, which is to promote the canonical positions and momenta to operators satisfying canonical commutation relations. To do this, we start by expressing the Hamiltonian directly in terms of the canonically conjugate variables which are $A_i, \Pi_i = -D_i$. This is done as follows: let us define the relationship $\mathbf{E}(\mathbf{D})$ via

$$\mathbf{E} = \sum_n \eta^{(n)} : \mathbf{D}^{\otimes n}. \tag{I.26}$$

The values of η are expressed in terms of the nonlinear susceptibilities χ by writing $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ and solving the resulting equation order by order in $\mathbf{E}$ (see footnote for example[4].)

To express the Hamiltonian density $\mathcal{H}'$ in terms of $\mathbf{D}$, we make use of the following identity[5]:

$$\int \mathbf{E} \cdot d\mathbf{D} = \frac{1}{2}\epsilon_0 \mathbf{E}^2 + \epsilon_0 \sum_{n=1}^{\infty} \frac{n}{n+1} \chi^{(n)} : \mathbf{E}^{\otimes(n+1)} = \epsilon_0 \sum_{n=1}^{\infty} \frac{n}{n+1} \epsilon^{(n)} : \mathbf{E}^{\otimes(n+1)}. \tag{I.29}$$

Using this result, and expressing $\mathbf{E}$ in terms of $\mathbf{D}$ and evaluating the resulting integral (via the same manipulations as in Footnote 4), one arrives at

$$H = \int d^3r \sum_{n=1}^{\infty} \eta^{(n)} : \mathbf{D}^{\otimes(n+1)} + \frac{(\nabla \times \mathbf{A})^2}{2\mu_0}, \tag{I.30}$$

where I have expressed $\mathbf{B}$ in terms of $\mathbf{A}$ to express the Hamiltonian fully in terms of $\mathbf{D}, \mathbf{A}$.

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[4] The results for the first two terms are, for example,

$$\eta^{(1)}_{ij} = \frac{1}{\epsilon_0}(1 + \chi^{(1)})^{-1}_{ij} \tag{I.27}$$

$$\eta^{(2)}_{ijk} = -\epsilon_0 \eta^{(1)}_{il} \chi^{(2)}_{lmn} \eta^{(1)}_{mj} \eta^{(1)}_{nk}. \tag{I.28}$$

We see, for example, that the second order response $\eta^{(2)}$ is proportional to $\chi^{(2)}$, but now contains some extra factors related to the linear response. The third-order response $\eta^{(3)}$ will similarly involve $\eta^{(1)}$ and $\eta^{(2)}$

[5] This identity is proved as follows: write $\int \mathbf{E} \cdot d\mathbf{D} = \epsilon_0 \int \mathbf{E} \cdot d\mathbf{E} + \int \mathbf{E} \cdot d\mathbf{P}$. The first term is $\frac{1}{2}\epsilon_0 \mathbf{E}^2$. The second term is integrated by noting that $d\mathbf{P} = \sum_n d\mathbf{P}^{(n)} = n\mathbf{P}^{(n)} \cdot d\mathbf{E}$. The factor of n follows from Kleinman symmetry. The remaining integral may be done by similarly noting that $\mathbf{P}^{(n)} \dot{d}\mathbf{E} = \frac{1}{n+1} d(\chi^{(n)} : \mathbf{E}^{\otimes(n+1)})$.

C. Quantization

The canonical quantization proceeds by promoting $\mathbf{A}$ and $\mathbf{D}$ to operators satisfying canonical commutation relations. Since we can think of fields as a collection of c numbers for each position and each direction, it is tempting to impose the commutation relation $[A_i(\mathbf{r}), \Pi_j(\mathbf{r}')] = [A_i(\mathbf{r}), -D_j(\mathbf{r}')] = i\hbar\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')$. However this cannot be right as it is not consistent with $\nabla\cdot\mathbf{A} = 0$ and $\nabla\cdot\mathbf{D} = 0$. In fact, since both fields are divergenceless, the resulting tensor function needs to project functions onto other divergenceless (transverse) functions. The corresponding answer is instead

$$[D_i(\mathbf{r}), A_j(\mathbf{r}')] = i\hbar\delta^{\perp}_{ij}(\mathbf{r}-\mathbf{r}'). \tag{I.31}$$

The object $\delta^{\perp}_{ij}(\mathbf{r}-\mathbf{r}')$ is called the *transverse delta function* and is a distribution defined by

$$\int d^3r'\ \delta^{\perp}_{ij}(\mathbf{r}-\mathbf{r}')X_j(\mathbf{r}') = X_i^{\perp}(\mathbf{r}), \tag{I.32}$$

where $\mathbf{X}^{\perp}(\mathbf{r})$ is the divergence-less (transverse) part of $\mathbf{X}$ defined such that $\nabla\cdot\mathbf{X}^{\perp} = 0$. Recall from electrodynamics that the Helmholtz-Hodge decomposition enables us to decompose an arbitrary differentiable vector function into a divergence-less (transverse) part and a curl-less (longitudinal) part[6]. We will take this commutator as the fundamental postulate of macroscopic quantum electrodynamics. With this, and the Hamiltonian, all else follows.

It will turn out that in many cases, we want commutators of the displacement field and the magnetic field. That commutator is given by:

$$[D_i(\mathbf{r}), B_j(\mathbf{r}')] = i\hbar\epsilon_{jlm}\partial'_l\delta^{\perp}_{im}(\mathbf{r}-\mathbf{r}'), \tag{I.34}$$

where ∂' denotes derivative with respect to $\mathbf{r}'$.

[6] The names transverse and longitudinal are most easily understood by noting that a divergence-less function, in Fourier space, satisfies $\mathbf{k}\cdot\mathbf{X}(\mathbf{k}) = 0$ while a curl-less function satisfies $\mathbf{k}\times\mathbf{X}(\mathbf{k}) = 0$. Therefore a transverse (longitudinal) function, in Fourier space, has the property that the vector field $\mathbf{X}(\mathbf{k})$ is transverse (longitudinal) to $\mathbf{k}$. From this reasoning, you can convince yourself that in Fourier space the transverse delta function is represented by

$$\delta^{\perp}_{ij}(\mathbf{r}-\mathbf{r}') = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}(\delta_{ij} - \hat{k}_i\hat{k}_j). \tag{I.33}$$

The tensor structure precisely projects out the part of a vector transverse to $\mathbf{k}$.

D. Heisenberg picture description of the quantized electromagnetic field

To check that this commutator and Hamiltonian are valid, we now derive the Heisenberg equations of motion and show that in the classical limit, they reduce exactly to Maxwell's equations in a nonlinear medium. Recall that the Heisenberg equations are generally expressed as

$$\dot{O} = \frac{i}{\hbar}[H, O], \tag{I.35}$$

where O is some operator. Let us start by examining the time-development of $\mathbf{D}$, which satisfies

$$\partial_t D_j(\mathbf{x}) = \frac{i}{\hbar}[H, D_j(\mathbf{x})] = \frac{i}{2\hbar\mu_0}\int d^3r\ [B_k(\mathbf{r})^2, D_j(\mathbf{x})], \tag{I.36}$$

which using the fundamental commutator can be written as

$$\partial_t D_j(\mathbf{x}) = \frac{i}{\hbar\mu_0}\int d^3r\ B_k(\mathbf{r})(-i\hbar\epsilon_{klm}\partial_l\delta^{\perp}_{jm}(\mathbf{r}-\mathbf{x})), \tag{I.37}$$

which can be written as

$$\partial_t D_j(\mathbf{x}) = \frac{1}{\mu_0}\int d^3r\ (\nabla\times\mathbf{B})_m\delta^{\perp}_{jm}(\mathbf{r}-\mathbf{x})) = \frac{1}{\mu_0}(\nabla\times\mathbf{B})_j(\mathbf{x}). \tag{I.38}$$

In other words, we have

$$\nabla\times\mathbf{H} = \partial_t\mathbf{D}, \tag{I.39}$$

which is the Ampere-Maxwell law in operator form.

Let us now derive an equation for the time-development of the magnetic field. That equation is

$$\partial_t B_j(\mathbf{x}) = \frac{i}{\hbar}[H, B_j(\mathbf{x})] = \frac{i}{\hbar}\int d^3r\ \sum_{n=1}^{\infty}\frac{1}{n+1}[\eta^{(n)} : \mathbf{D}(\mathbf{r})^{\otimes(n+1)}, B_j(\mathbf{x})]. \tag{I.40}$$

Using the product rule for commutators (e.g., $[AB, C] = A[B, C] + [A, C]B$) repeatedly, the above reduces to

$$\partial_t B_j(\mathbf{x}) = \frac{i}{\hbar}\int d^3r\ \sum_{n=1}^{\infty}(\eta^{(n)} : \mathbf{D}(\mathbf{r})^{\otimes n})_{i_0}\times i\hbar\epsilon_{jlm}\partial'_l\delta^{\perp}_{i_0m}(\mathbf{r}-\mathbf{x}), \tag{I.41}$$

where $(\eta^{(n)} : \mathbf{D}(\mathbf{r})^{\otimes n})_{i_0} = \eta^{(n)}_{i_0i_1\cdots i_n}D_{i_1}\cdots D_{i_n}$ and the primed derivative denotes differentiation with respect to $\mathbf{x}$. To simplify this further, we note that since

$E_{i_0} = \sum_{n=1}^{\infty} (\eta^{(n)} : \mathbf{D}(\mathbf{r})^{\otimes n})_{i_0}$, we my write

$$\partial_t B_j(\mathbf{x}) = \frac{i}{\hbar} \int d^3r \; E_{i_0}(\mathbf{r}) \times i\hbar \epsilon_{jlm} \partial_l' \delta^{\perp}_{i_0 m}(\mathbf{r} - \mathbf{x}). \tag{I.42}$$

To proceed, we write

$$\partial_t B_j(\mathbf{x}) = -\epsilon_{jlm} \partial_l' \int d^3r \; E_{i_0}(\mathbf{r}) \times \delta^{\perp}_{i_0 m}(\mathbf{r} - \mathbf{x}) = -\epsilon_{jlm} \partial_l' E^{\perp}_m(\mathbf{x}) = -(\nabla \times \mathbf{E}^{\perp})_j(\mathbf{x}). \tag{I.43}$$

Note however that $\nabla \times \mathbf{E}^{\perp} = \nabla \times \mathbf{E}$ since $\nabla \times \mathbf{E}^{||} = 0$ by definition. Therefore, we simply have

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}. \tag{I.44}$$

This corresponds to Faraday's law in operator form.

We should mention that the electric and magnetic Gauss' laws follow as constraints in the quantization and were effectively imposed by setting the commutator of the displacement and magnetic field to be a transverse delta function. In the classical limit, we replace all operators with commuting numbers and we recover the classical Maxwell equations, suggesting the validity of our canonical quantization. We now move to study the eigenstates of the Hamiltonian, which are key to understanding quantum optics.

E. Photons in a medium

Let us consider what happens in one of the simplest possible cases: a linear, homogeneous, and isotropic medium. In this case, we can write the Maxwell equations for the electric fields as

$$\nabla^2 \mathbf{E} - \frac{n^2}{c^2} \partial_t^2 \mathbf{E} = 0. \tag{I.45}$$

It will be helpful to write the electric field as

$$\mathbf{E}(\mathbf{r}, t) = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda} E_{\lambda}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{r}} \hat{\epsilon}_{\lambda}(\mathbf{k}), \tag{I.46}$$

where $\hat{\epsilon}_{\lambda}(\mathbf{k})$ is a polarization vector satisfying $\mathbf{k} \cdot \hat{\epsilon}_{\lambda}(\mathbf{k}) = 0$. In other words, we are decomposing the electric field into a sum of transverse plane waves. We refer to this representation of the electric field as a “mode expansion”. Now, recall that $\mathbf{E}$ is an operator, and it is an operator which represents a physically observable quantity and must therefore be Hermitian. That imposes a constraint:

$$E_\lambda(\mathbf{k},t)\hat{\epsilon}_\lambda(\mathbf{k}) = E_\lambda(-\mathbf{k},t)^\dagger\hat{\epsilon}^*_\lambda(-\mathbf{k}). \tag{I.47}$$

Note that the two-dimensional vector space spanned by the polarization vectors for $\mathbf{k}$ and $-\mathbf{k}$ is identical, and so we are free to impose for example that $\hat{\epsilon}_\lambda(\mathbf{k}) = \hat{\epsilon}^*_\lambda(-\mathbf{k})$. We can also impose orthonormality on the two polarization vectors for any given $\mathbf{k}$ such that $\hat{\epsilon}^*_\lambda(\mathbf{k})\cdot\hat{\epsilon}_{\lambda'}(\mathbf{k}) = \delta_{\lambda\lambda'}$ (with $\lambda = 1,2$). Therefore, we may simply say that Hermiticity requires

$$E_\lambda(\mathbf{k},t) = E_\lambda(-\mathbf{k},t)^\dagger. \tag{I.48}$$

Plugging our mode expansion into the operator Maxwell wave equation yields

$$\ddot{E}_\lambda(\mathbf{k},t) + \omega_k^2 E_\lambda(\mathbf{k},t) = 0, \tag{I.49}$$

with $\omega_k = ck/n$. The corresponding oscillator equation is solved by complex exponentials via

$$E_\lambda(\mathbf{k},t) = a^E_\lambda(\mathbf{k})e^{-i\omega_k t} + b^E_\lambda(\mathbf{k})e^{i\omega_k t}. \tag{I.50}$$

The Hermiticity condition on $\mathbf{E}$ implies that

$$a^E_\lambda(\mathbf{k}) = b^E_\lambda(-\mathbf{k})^\dagger. \tag{I.51}$$

Therefore, we may write our mode expansion for the electric field operator as

$$\mathbf{E}(\mathbf{r},t) = \int \frac{d^3k}{(2\pi)^3}\sum_\lambda \left(a^E_\lambda(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_k t}\hat{\epsilon}_\lambda(\mathbf{k}) + a^E_\lambda(\mathbf{k})^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_k t}\hat{\epsilon}^*_\lambda(\mathbf{k})\right). \tag{I.52}$$

As of now, the $a_\lambda(\mathbf{k})$ operators are ostensibly arbitrary operator-valued constants. They however are not arbitrary as the corresponding $\mathbf{D}$ and $\mathbf{A}$ fields implied by this expression need to satisfy the canonical commutation relations. The $\mathbf{A}$ field is given by

$$\mathbf{A}(\mathbf{r},t) = \int \frac{d^3k}{(2\pi)^3}\sum_\lambda \frac{1}{i\omega_k}\left(a^E_\lambda(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_k t}\hat{\epsilon}_\lambda(\mathbf{k}) - a^E_\lambda(\mathbf{k})^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_k t}\hat{\epsilon}^*_\lambda(\mathbf{k})\right), \tag{I.53}$$

while the displacement field is given by

$$\mathbf{D}(\mathbf{r},t) = \int \frac{d^3k}{(2\pi)^3}\sum_\lambda \epsilon_0\epsilon\left(a^E_\lambda(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_k t}\hat{\epsilon}_\lambda(\mathbf{k}) + a^E_\lambda(\mathbf{k})^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_k t}\hat{\epsilon}^*_\lambda(\mathbf{k})\right). \tag{I.54}$$

To evaluate the commutator $[D_i(\mathbf{r}), A_j(\mathbf{r}')]$, we will need to evaluate the commutator $[a^E_\lambda(\mathbf{k}), a^E_{\lambda'}(\mathbf{k}')^\dagger]$. To converge on final notation, let us define $a^E_\lambda(\mathbf{k}) = i\mathcal{E}_\lambda(\mathbf{k})a_\lambda(\mathbf{k})$[7]. It is clear that from the standpoint of the electromagnetic field, different wavevectors are independent of each other and different polarizations are as well.

Therefore, we impose the commutation relation

$$[a^E_\lambda(\mathbf{k}), a^E_{\lambda'}(\mathbf{k}')^\dagger] \equiv |\mathcal{E}_\lambda(\mathbf{k})|^2[a_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k}')] = |\mathcal{E}_\lambda(\mathbf{k})|^2 \times (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\delta_{\lambda,\lambda'}. \quad \text{(I.55)}$$

Using these results, the equal-time[8] canonical commutator is found to be

$$[D_i(\mathbf{r},t), A_j(\mathbf{r},t)] = \int \frac{d^3k}{(2\pi)^3}\sum_\lambda |\mathcal{E}_\lambda(\mathbf{k})|^2 \frac{i\epsilon_0\epsilon}{\omega_k}\left(e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}\hat{\epsilon}_{\lambda,i}(\mathbf{k})\hat{\epsilon}^*_{\lambda,j}(\mathbf{k}) + e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')}\hat{\epsilon}^*_{\lambda,i}(\mathbf{k})\hat{\epsilon}_{\lambda,j}(\mathbf{k})\right). \quad \text{(I.56)}$$

If we take $|\mathcal{E}_\lambda(\mathbf{k})|^2 = \frac{\hbar\omega_k}{2\epsilon_0\epsilon}$, we can perform the sum over polarizations and find[9]

$$[D_i(\mathbf{r},t), A_j(\mathbf{r},t)] = \frac{1}{2}i\hbar\int\frac{d^3k}{(2\pi)^3}(e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')} + e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')})(\delta_{ij} - \hat{k}_i\hat{k}_j). \quad \text{(I.57)}$$

Taking $\mathbf{k}\to-\mathbf{k}$ in the second term immediately yields

$$[D_i(\mathbf{r},t), A_j(\mathbf{r},t)] = i\hbar\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}(\delta_{ij} - \hat{k}_i\hat{k}_j) = i\hbar\delta^\perp_{ij}(\mathbf{r}-\mathbf{r}'). \quad \text{(I.58)}$$

Summarizing, we may write our electric field operator as

$$\mathbf{E}(\mathbf{r},t) = i\int\frac{d^3k}{(2\pi)^3}\sum_\lambda\sqrt{\frac{\hbar\omega_k}{2\epsilon_0\epsilon}}\left(a_\lambda(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_k t}\hat{\epsilon}_\lambda(\mathbf{k}) - a_\lambda(\mathbf{k})^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_k t}\hat{\epsilon}^*_\lambda(\mathbf{k})\right). \quad \text{(I.59)}$$

The other operators follow immediately from the Maxwell equations as

$$\mathbf{D}(\mathbf{r},t) = i\int\frac{d^3k}{(2\pi)^3}\sum_\lambda\sqrt{\frac{\epsilon_0\epsilon\hbar\omega_k}{2}}\left(a_\lambda(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_k t}\hat{\epsilon}_\lambda(\mathbf{k}) - a_\lambda(\mathbf{k})^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_k t}\hat{\epsilon}^*_\lambda(\mathbf{k})\right) \quad \text{(I.60)}$$

and

$$\mathbf{A}(\mathbf{r},t) = \int\frac{d^3k}{(2\pi)^3}\sum_\lambda\sqrt{\frac{\hbar}{2\epsilon_0\epsilon\omega_k}}\left(a_\lambda(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_k t}\hat{\epsilon}_\lambda(\mathbf{k}) + a_\lambda(\mathbf{k})^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_k t}\hat{\epsilon}^*_\lambda(\mathbf{k})\right). \quad \text{(I.61)}$$

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[7] The factor of i is arbitrary of course as we could have chosen any phase: we just choose the phase convention that aligns with standard texts on non-relativistic quantum electrodynamics.

[8] The commutators we have been writing thus far are Schrodinger picture operator commutators. In the Heisenberg picture, so long as the operators are evaluated at the same time, it is equal to the Schrodinger picture (t=0) commutator.

[9] I've used the fact that the two polarization vectors and $\hat{k}$ span three-dimensional space such that $\hat{k}_i\hat{k}_j + \sum_\lambda \hat{\epsilon}_{\lambda,i}(\mathbf{k})\hat{\epsilon}^*_{\lambda,j}(\mathbf{k}) = \delta_{ij}$.

Let us use these results to express the Hamiltonian in terms of $a_\lambda(\mathbf{k})$ and $a_\lambda(\mathbf{k})^\dagger$. The result can be quickly found to be

$$H = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \frac{\hbar\omega_k}{2} \left(a_\lambda(\mathbf{k}) a_\lambda(\mathbf{k})^\dagger + a_\lambda(\mathbf{k})^\dagger a_\lambda(\mathbf{k}) \right) \tag{I.62}$$

Using the commutation relations for $a, a^\dagger$, we have

$$H = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \hbar\omega_k a(\mathbf{k})^\dagger a(\mathbf{k}) + H_{\text{Z.P.}}, \tag{I.63}$$

where $H_{\text{Z.P.}}$ is an infinite constant called the zero-point energy. It has no effect on physical predictions that we will make. Thus, in what follows, we will drop the constant and write

$$H = \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \hbar\omega_k a(\mathbf{k})^\dagger a(\mathbf{k}). \tag{I.64}$$

Now, the commutation relations for $a, a^\dagger$ are clearly those of creation and annihilation operators of the harmonic oscillator. This should be no surprise as we essentially introduced these operators to diagonalize a quadratic Hamiltonian. The electromagnetic field Hamiltonian is essentially the field theory version of $p^2/2m + m\omega^2 x^2/2$.

F. Box quantization

While everything thus far has been rigorous, it is somewhat inconvenient to normalize $a, a^\dagger$ such that their commutation relations are delta functions. Instead, we consider our continuous space as the limit of a large box of volume V with periodic boundaries as $V \to \infty$. The periodic boundary conditions impose that $\mathbf{k} = \frac{2\pi}{L}(m_x, m_y, m_z)$ where $m_{x,y,z}$ are integers. Then, our integrals over wavevectors are replaced by discrete sums and our delta-function commutators are replaced by unit commutators. Noting that $\sum_k = \frac{V}{(2\pi)^3} \int d^3k$, and that $\delta(\mathbf{k} = 0) = \lim_{V \to 0} V/(2\pi)^3$, we may define a dimensionless $a_\mathbf{k} = a(\mathbf{k})/\sqrt{V}$ which has the property that $[a_{\mathbf{k},\lambda}, a^\dagger_{\mathbf{k}',\lambda'}] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}$. Then, we may write

$$H = \sum_{\mathbf{k},\lambda} \hbar\omega_k a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} + H_{\text{Z.P.}}, \tag{I.65}$$

with $[a_{\mathbf{k},\lambda}, a^\dagger_{\mathbf{k}',\lambda'}] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}$. Field operators may be written in the following way. The electric field operator can be written as

$$\mathbf{E}(\mathbf{r},t) = i\sum_{\mathbf{k},\lambda}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0\epsilon V}}\left(a_{\mathbf{k},\lambda}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_k t}\hat{\epsilon}_{\mathbf{k},\lambda} - a^\dagger_{\mathbf{k},\lambda}e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_k t}\hat{\epsilon}^*_{\mathbf{k},\lambda}\right). \tag{I.66}$$

II. QUANTUM STATES OF LIGHT

In the previous section, we showed that we could describe the electromagnetic field the way we would describe a collection of independent harmonic oscillators. There is one oscillator for each classical solution of Maxwell's equations, which in free space, is labeled by a wavevector and a polarization transverse to the wavevector. The electromagnetic field operators, specifically the vector potential and (minus) displacement field act as position and momentum-like variables and they are expressed in terms of linear combinations of creation and annihilation operators. The Hamiltonian, or energy, is also expressed in the way we expect for a collection of simple harmonic oscillators. Thus, we can now very easily talk about a basis of wavefunctions which span the Hilbert space describing electromagnetic field states. That Hilbert space, $\mathbb{H}_{\mathrm{EM}}$ could be written as

$$\mathbb{H}_{\mathrm{EM}} = \mathrm{span}\{|n_{\mathbf{k}_1,\hat{\epsilon}_{\mathbf{k}_1}}, n_{\mathbf{k}_2,\hat{\epsilon}_{\mathbf{k}_2}}, \ldots\rangle\} \equiv \mathrm{span}\left\{\bigotimes_{\mathbf{k},\hat{\epsilon}_\mathbf{k}}|n_{\mathbf{k},\hat{\epsilon}_\mathbf{k}}\rangle\right\}, \tag{II.1}$$

where $\otimes$ is the usual tensor product which allows us to adjoin multiple Hilbert spaces and the state $|n\rangle$ is a *number state*, *Fock state*, or simply, an *n-photon state*, defined via

$$|n\rangle = \frac{a^\dagger}{\sqrt{n}}|0\rangle, \tag{II.2}$$

with $|0\rangle$, the *zero-photon state* or *vacuum state* defined such that

$$a|0\rangle = 0. \tag{II.3}$$

The label $\mathbf{k}, \hat{\epsilon}_\mathbf{k}$ on $n_{\mathbf{k},\hat{\epsilon}_\mathbf{k}}$ denotes that we are taking about an n-photon state in a field mode with wavevector $\mathbf{k}$ and polarization $\hat{\epsilon}_\mathbf{k}$.

The notation is more complicated than the physics! The state $|n_{\mathbf{k}_1,\hat{\epsilon}_{\mathbf{k}_1}}, n_{\mathbf{k}_2,\hat{\epsilon}_{\mathbf{k}_2}}, \ldots\rangle$ just represents a state in which there are $n_{\mathbf{k}_1,\hat{\epsilon}_{\mathbf{k}_1}}$ photons in $\mathbf{k}_1, \hat{\epsilon}_{\mathbf{k}_1}$, $n_{\mathbf{k}_2,\hat{\epsilon}_{\mathbf{k}_2}}$ photons in $\mathbf{k}_2, \hat{\epsilon}_{\mathbf{k}_2}$, $n_{\mathbf{k}_3,\hat{\epsilon}_{\mathbf{k}_3}}$ photons in $\mathbf{k}_3, \hat{\epsilon}_{\mathbf{k}_3}$ and so on. What the notation does however make abundantly clear is that the state space in quantum electrodynamics is vast in comparison with that of classical electrodynamics. In classical EM, the field is specified by one complex number for each wavevector-polarization pair. In quantum EM, each wavevector-polarization pair has an infinite-dimensional vector space attached to it! This much larger space of possibilities, directly leads to a number of behaviors of the EM field which are impossible to enable in classical EM. These novel behaviors typically concern the *statistical properties* of light. We will explore that in the following sections. To do this, let us take an *very* simple approach which is to consider an *single* mode of the electromagnetic field. Here, you can imagine a mode to be that which labels a classical solution of source-free Maxwell. In free-space, that is a wavevector-polarization pair. Other examples of single modes can be light confined in an optical cavity at a fixed resonance frequency. Although this may appear restrictive, many physically relevant situations—such as a Gaussian beam interacting with a nonlinear crystal—and sometimes even optical pulses, can be well understood at the qualitative level using the single-mode description.

A. Single-mode quantum states of light

A single mode of light is described by a quantum harmonic oscillator with Hamiltonian

$$H = \hbar\omega_0 a^\dagger a, \tag{II.4}$$

with $[a, a^\dagger] = 1$. The field operators in this case can be written for example as

$$\mathbf{E}(\mathbf{r}, t) = \mathcal{E}(\mathbf{r}) a e^{-i\omega t} + \mathcal{E}^*(\mathbf{r}) a^\dagger e^{i\omega t}. \tag{II.5}$$

For free space, $\mathcal{E}(\mathbf{r}) = \sqrt{\frac{\hbar\omega}{2\epsilon V}} e^{i\mathbf{k}\cdot\mathbf{r}} \hat{\epsilon}_{\mathbf{k}}$. For more complicated structured media, the $\mathcal{E}(\mathbf{r})$ satisfies the appropriate source-free Maxwell equation. We see that for a given position, the vector potential is proportional to $X \equiv (a + a^\dagger)/\sqrt{2}$ electric field is proportional to $P = i(a^\dagger - a)/\sqrt{2}$, which are position and momentum like operators of an oscillator. We refer to X, P as *quadratures* of the electromagnetic field. They satisfy an uncertainty principle $\Delta X \Delta P = i$.

1. Fock states

We start our discussion of quantum states of light by analyzing simplest states from the Hamiltonian perspective, which are the eigenstates, called Fock states, or number states, or n-photon states. As you recall from your studies of the quantum harmonic oscillator[10] the states $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}$ are eigenstates of the Hamiltonian with energy $E = n\hbar\omega$ (up to the overall constant). We can see why they're called n-photon states: we expect from our earliest interactions with wave-particle duality that n photons will have energy $n\hbar\omega$ and we see that these n-photon states are labeled by non-negative integers. For this reason, we also call the operator $N = a^\dagger a$ the photon number operator, as its eigenstates are n-photon states $|n\rangle$ with eigenvalue n.

To get a better sense of these states, let us discuss expectation values of various operators:

[10] This footnote provides a short refresher on oscillator eigenstates. The quantum harmonic oscillator admits a basis of states with well-defined photon number, known as Fock states, denoted $|n\rangle$. These states have an exactly defined number of photons in the mode. For example, $|0\rangle$ contains zero photons and is referred to as the vacuum state, while $|1\rangle$ is a state containing a single photon. The creation and annihilation operators act on these Fock states by creating and destroying photons, with the algebraic rules:

$$a\,|n\rangle = \sqrt{n}\;|n-1\rangle \tag{II.6}$$

$$a^\dagger\,|n\rangle = \sqrt{n+1}\;|n+1\rangle\,. \tag{II.7}$$

For example, the single photon Fock state is $|1\rangle = a^\dagger\,|0\rangle$. Higher order Fock states can be created by applying the creation operator multiple times. As a result, a Fock state with n photons is generally given by

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle\,. \tag{II.8}$$

Fock states are often discussed in the context of the number operator:

$$N \equiv a^\dagger a. \tag{II.9}$$

Specifically, Fock states $|n\rangle$ are eigenstates of n

$$N\,|n\rangle = n\,|n\rangle\,, \quad n = 0, 1, 2, \ldots \tag{II.10}$$

They also form an orthonormal basis for the so-called "Hilbert space" of a quantum oscillator mode:

$$\langle m|n\rangle = \delta_{mn}, \qquad \sum_{n=0}^{\infty} |n\rangle\,\langle n| = I. \tag{II.11}$$

1. The mean photon number $\langle N \rangle = \langle n|a^\dagger a|n\rangle = n\langle n|n\rangle = n$.

2. The variance in the photon number in the Fock state $|n\rangle$ is $(\Delta N)^2 = \langle N^2\rangle - \langle n\rangle^2 = 0$. This follows from the fact that for any operator function $f(N)$ we have $f(N)|n\rangle = f(n)$ and so $\langle N^2\rangle = n^2$.

3. The mean electric field at any point, $\langle E(\mathbf{r},t)\rangle$, is zero since $\langle n|a|n\rangle = \langle n|a^\dagger|n\rangle^* = 0$. Similarly, since the magnetic field is also a linear combination of $a, a^\dagger$ its mean is also zero.

4. The variance in any component of the electric field is non-zero. Since the mean vanishes, we have $(\Delta E_i(\mathbf{r},t))^2 = \langle E_i^2(\mathbf{r},t)\rangle = \langle n| \left(\mathcal{E}_i^2(\mathbf{r})a^2 e^{-2i\omega t} + |\mathcal{E}_i(\mathbf{r})|^2(aa^\dagger + a^\dagger a) + \mathcal{E}_i^{*2}(\mathbf{r})a^{\dagger 2}e^{2i\omega t}\right)|n\rangle$. The terms with $a^2, a^{\dagger 2}$ give no contribution. The remain terms can be evaluated using $aa^\dagger + a^\dagger a = 2N+1$, leaving us with $(\Delta E_i(\mathbf{r},t))^2 = (2n+1)|\mathcal{E}_i(\mathbf{r})|^2$.

5. It is useful to also talk about the quadrature variances. It is easy to see that $(\Delta X)^2 = (\Delta P)^2 = n + \frac{1}{2}$.

The first two properties are expected from the name: an n-photon state should have a well-defined number of photons n and so there should be zero variance around the mean n. The electric field is less intuitive, and tells you already that the Fock state is not like the lasers we work with in optics typically, that have a well-defined average, sinusoidally oscillating field. The average field is zero, but its square is nonzero on average - representing a type of light with a random electric field [11].

Importantly, the field variance is nonzero even for the ground state of the harmonic oscillator, which has no photons in it! This nonzero variance in the absence of photons is known as vacuum fluctuations (or zero-point fluctuations). It reflects the fact that the electric field, like position in a harmonic oscillator, cannot be sharply defined

[11] For the n photon state, these properties can be understood by analogy to the harmonic oscillator in the coordinate representation. We know that the electric field operator is like the p operator, and so we can talk about the momentum wavefunctions of the oscillator eigenstates. You may be less familiar with the momentum wavefunctions of the eigenstates, but they are the same as the position wavefunctions since the Hermite-Gaussian polynomials are their own Fourier transform. Therefore, we can say there is a wavefunction whose square gives the probability of measuring the electric field at each point and it is given by a Hermite-Gaussian.

in the ground state due to the uncertainty principle. We highlight that this is a purely quantum phenomenon with no classical analog: in classical electromagnetism, a field with zero amplitude has no fluctuations and carries no energy. In contrast, the quantum vacuum is an active state, with inevitable fluctuations which arise from quantum mechanical uncertainty. These vacuum fluctuations play a central role in many phenomena in quantum and nonlinear optics. For example, they act as the seed for spontaneous processes such as spontaneous emission and parametric down-conversion, and set fundamental noise limits in optical measurements. More broadly, they provide the background upon which nonlinear interactions generate nonclassical states of light, such as squeezed and entangled states, which we will study in later sections.

B. Coherent states

Let us now discuss a class of states which are in some sense, the "most classical states" of the electromagnetic field. These states have a well-defined expectation value of the electromagnetic field which satisfies the classical Maxwell equations, and in the limit of a large enough number of photons, has minimal relative fluctuations of the field relative to the average. In other words, they represent a classical electromagnetic field with a small amount of quantum noise on top, which can be thought of as vacuum fluctuations. These states are called *coherent states*, are labeled $|\alpha\rangle$ and can be defined in the following way:

$$|\alpha\rangle = D(\alpha)|0\rangle \equiv e^{\alpha a^\dagger - \alpha^* a}|0\rangle, \tag{II.12}$$

where $D(\alpha)$ is a unitary operator called the *displacement operator*. A useful way to calculate expectation values is in terms of a Heisenberg picture. Although we are not explicitly referring to time, since $D(\alpha)$ is unitary, we can talk about unitary as affecting a corresponding operator transformation

$$a \to D^\dagger(\alpha) a D(\alpha). \tag{II.13}$$

The most standard way to evaluate this is using the so-called Campbell identity[12]. The Campbell identity applied in this context,with $B = a, A = \alpha^* a - \alpha a^\dagger$, states that

$$D^\dagger(\alpha) a D(\alpha) = a + \alpha. \tag{II.15}$$

All higher-order terms in the series vanish. We see that the displacement operator indeed displaces the annihilation operator. Let us now ask about some of the expectation values we discussed before, but in reverse order. Let's talk about fields, and then photon numbers. Let us also ask about what happens when the coherent state freely evolves in time. If we have a coherent state at time $t = 0$, then at time t, the resulting state is

$$|\alpha(t)\rangle = e^{-iHt/\hbar}|\alpha\rangle = e^{-i\omega t N}|\alpha\rangle = e^{-i\omega t N} D(\alpha)|0\rangle. \tag{II.16}$$

To evaluate this, we multiply by one:

$$|\alpha(t)\rangle = e^{-iHt/\hbar}|\alpha\rangle = e^{-i\omega t N} D(\alpha) e^{i\omega t N} e^{-i\omega t N}|0\rangle. \tag{II.17}$$

Using $e^{-i\omega t N} a e^{i\omega t N} = a e^{-i\omega t}$ and $N|0\rangle = 0$, we may write[13]

$$|\alpha(t)\rangle = D(\alpha e^{-i\omega t})|0\rangle. \tag{II.18}$$

Therefore the operator at time t may be written as

$$a(t) = a + \alpha e^{-i\omega t}. \tag{II.19}$$

With this we may now straightforwardly compute various expectation values. Let's start with the ith component of the field. In this Heisenberg picture, this is computed

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[12] The Campbell identity is a useful formula often used to evaluate unitary transformations in quantum mechanics. Its statement is: consider two operators A and B and consider the quantity $e^A B e^{-A}$. Then it is given by

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots. \tag{II.14}$$

A corollary of this identity is that if $[A, B] = cI$ with I the identity, then $e^A B e^{-A} = B + c$.

[13] As a reminder, in the Heisenberg picture, the free evolution of a, given by $a(t) = U^\dagger a U$ with $U = e^{-iHt/\hbar}$ is given by $\dot{a} = \frac{i}{\hbar}[\hbar\omega a^\dagger a, a] = -i\omega a \implies a(t) = e^{-i\omega t}a(0)$. Further, for any operator like $D(\alpha)$, we can see that by expanding it in a Taylor series: $D(\alpha) = \sum_n \frac{1}{n!}(\alpha a^\dagger - \alpha^* a)^n$, the Heisenberg evolution of such an operator just amounts to replacing $a \to a(t)$ and $a^\dagger \to a^\dagger(t)$.

as

$$\langle E_i(r,t)\rangle = \langle \mathcal{E}(\mathbf{r})a(t) + \mathcal{E}^*(\mathbf{r})a^\dagger(t)\rangle = \langle 0|\mathcal{E}(\mathbf{r})a(t) + \mathcal{E}^*(\mathbf{r})a^\dagger(t)|0\rangle. \tag{II.20}$$

Importantly, the expectation value is taken with respect to the initial state, which can be taken as zero if we treat $a(t)$ as resulting from the combination of displacement and unitary time evolution. Therefore the answer is immediately found to be

$$\mathcal{E}(\mathbf{r})\alpha(t)e^{-i\omega t} + \mathcal{E}^*(\mathbf{r})a^* e^{i\omega t}. \tag{II.21}$$

This is finite and corresponds to a sinusoidal oscillation of the field, like you might imagine classically. The variance is nonzero, and is given by

$$(\Delta E_i(r,t))^2 = |\mathcal{E}(\mathbf{r})|^2, \tag{II.22}$$

which is is time-independent, and exactly the same value as in the vacuum state. Similarly to the Fock states, it is helpful to talk about the quadrature variances, which are easily found as $(\Delta X)^2 = (\Delta P)^2 = 1/2$.

The interpretation here is that the coherent state is a sinusoidal field with small vacuum or zero-point fluctuations. It may be intuitively thought of as having the least amount of noise possible (but this is not exactly right, as it is possible to find states with field variances below the value mentioned here).

1. Schrodinger picture description of coherent states

Let us also discuss the coherent state in its Fock representation (or equivalently, its energy eigenbasis representation).

We saw previously that Fock states have peculiar statistical properties (such as zero average electric field) which do not connect well to our classical conception of electromagnetism. It is thus natural to wonder about what quantum state of light might behave like a classical optical field. Such states are known as "coherent states," which are the topic of this section.

The coherent state can be seen as resulting from a unitary transformation of the vacuum state. We are then tasked with evaluating

$$|\alpha\rangle = D(\alpha)|0\rangle \equiv e^{\alpha a^\dagger - \alpha^* a}|0\rangle. \tag{II.23}$$

The simplest way to do this is to note that the transformation $D^\dagger(\alpha)aD(\alpha) = a + \alpha$ implies that the coherent state is an eigenstate of the annihilation operator. To see this, consider

$$a|\alpha\rangle = DD^\dagger aD|0\rangle = D(a+\alpha)|0\rangle = \alpha|\alpha\rangle. \tag{II.24}$$

If we express an eigenstate of the annihilation operator in the energy / photon number eigenbasis, as

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle, \tag{II.25}$$

then the eigenstate condition implies

$$a|\alpha\rangle = \sum_{n=0}^{\infty} c_n\sqrt{n}|n-1\rangle = \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}|n\rangle = \alpha\sum_{n=0}^{\infty} c_n|n\rangle, \tag{II.26}$$

implying the recurrence relation $c_{n+1} = \frac{\alpha}{\sqrt{n+1}}c_n$.Taking $c_0 = \mathcal{N}$ with $\mathcal{N}$ a normalization constant implies

$$|\alpha\rangle = \mathcal{N}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{II.27}$$

The normalization condition is $|\mathcal{N}|^2\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!} = 1$ which implies $|\mathcal{N}| = e^{-\frac{1}{2}|\alpha|^2}$. This allows us to write the coherent state in the Fock basis as

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{II.28}$$

Importantly, the coherent state is not an eigenstate of the Hamiltonian and is a superposition of photon number states. The probability of finding the system to have n photons is

$$P(n) = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}, \tag{II.29}$$

which is a Poisson distribution of mean $\langle n\rangle = |\alpha|^2$. The variance of the photon number is equal to the mean (and all cumulants of the photon number distribution generally are equal).

C. Squeezed states

We saw in the previous section that coherent states exhibit vacuum fluctuations: while they have a nonzero mean electric field, their fluctuations are the same as those of the vacuum. In particular, the uncertainty principle constrains the product of quadrature variances, fixing a lower bound on how small the fluctuations can be. This naturally raises the question of whether it is possible to reduce the fluctuations in one quadrature below the vacuum level, at the expense of increased fluctuations in the conjugate quadrature. The class of states known as squeezed states exhibits precisely this behavior.

To introduce squeezed states, we define the single-mode squeezing operator:

$$S(\zeta) = \exp\left(\frac{1}{2}\zeta^* a^2 - \frac{1}{2}\zeta(a^\dagger)^2\right), \tag{II.30}$$

where

$$\zeta = re^{i\phi} \tag{II.31}$$

is the complex squeezing parameter. While this may seem unmotivated at the moment, we note that this form of the squeezing operator arises naturally from common interactions in nonlinear optics, such as parametric amplification in $\chi^{(2)}$ media or four-wave mixing in $\chi^{(3)}$ systems, where pairs of photons are created or annihilated. We will show this in greater detail later.

A state known as the “squeezed vacuum” state is obtained by applying $S(\zeta)$ to the vacuum:

$$|\zeta\rangle = S(\zeta)\,|0\rangle\,. \tag{II.32}$$

A key property is that $S(\zeta)$ performs a Bogoliubov transformation on the operators:

$$S^\dagger(\zeta)\,a\,S(\zeta) = a\cosh r - e^{i\phi}a^\dagger\sinh r, \tag{II.33}$$

$$S^\dagger(\zeta)\,a^\dagger\,S(\zeta) = a^\dagger\cosh r - e^{-i\phi}a\sinh r. \tag{II.34}$$

In what follows, we will write the transformation as

$$a \to \mu a + \nu a^\dagger, \tag{II.35}$$

where $\mu = \cosh r, \nu = -e^{i\phi}$.

We now compute basic expectation values in the squeezed vacuum state. First, the mean field vanishes:

$$\langle a\rangle = \langle\zeta|\,a\,|\zeta\rangle \tag{II.36}$$

$$= \langle 0|\,S^\dagger(\zeta)\,a\,S(\zeta)\,|0\rangle \tag{II.37}$$

$$= \langle 0|\,\mu a + \nu a^\dagger\,|0\rangle = 0, \tag{II.38}$$

and similarly

$$\langle a^\dagger \rangle = 0. \tag{II.39}$$

This implies that any quadrature has mean value zero and therefore the electric and magnetic fields are also on average zero.

The mean photon number can be computed by explicitly using $|\zeta\rangle = S(\zeta)\,|0\rangle$:

$$\begin{aligned}
\langle n \rangle &= \langle \zeta |\, a^\dagger a \,| \zeta \rangle && \text{(II.40)}\\
&= \langle 0 |\, S^\dagger(\zeta)\, a^\dagger a\, S(\zeta)\,|0\rangle && \text{(II.41)}\\
&= \langle 0 |\, (\mu^* a^\dagger + \nu^* a)(\mu a + \nu a^\dagger)\,|0\rangle && \text{(II.42)}\\
&= |\nu|^2 = \sinh^2 r. && \text{(II.43)}
\end{aligned}$$

This is not zero, which is to be expected since the squeezed state is not a Hamiltonian eigenstate and thus must be a superposition of different photon number states. When $r \gg 1$, we see that the number of photons goes like $e^{2r}/4$ indicating a type of exponential gain[14].

Let us now look at the quadrature variances associated with the squeezed state. The variance

$$\begin{aligned}
(\Delta X_\theta)^2 &= \frac{1}{2}\langle 0 |\, \left((\mu a + \nu a^\dagger) e^{-i\theta} + (\mu^* a^\dagger + \nu^* a) e^{i\theta} \right)^2 |0\rangle && \text{(II.44)}\\
&= \frac{1}{2}\langle 0 |\, \left((\mu e^{-i\theta} + \nu^* e^{i\theta}) a + (\mu^* e^{i\theta} + \nu e^{-i\theta}) a^\dagger \right)^2 |0\rangle && \text{(II.45)}\\
&= \frac{1}{2}\left| \mu e^{-i\theta} + \nu^* e^{i\theta} \right|^2 && \text{(II.46)}\\
&= \frac{1}{2}\left| \cosh r e^{-i\theta} - \sinh r e^{i\theta - i\phi} \right|^2 && \text{(II.47)}\\
&= \frac{1}{2}\left| \cosh r e^{-i\psi} - \sinh r e^{i\psi} \right|^2, && \text{(II.48)}
\end{aligned}$$

where I have defined $\psi = \theta - \phi/2$.

[14] It is no accident that this looks very similar to the intensity of what is generated by a parametric amplifier. In fact, you may remember when we studied the degenerate parametric amplifier, we found a classical field transformation of the form $A(0) \to A(0)\cosh r + A^*(0) e^{i\phi} \sinh r$. We also found that the intensity for large r goes like $e^{2r}/4$. The Bogoliubov transformation is exactly the linear operator transformation corresponding to this classical transformation and it would therefore should not surprise you that the way to produce a squeezed state is by sending vacuum into a degenerate parametric amplifier!

Let us consider the quadrature angle θ that maximizes and minimizes the variance. Since the complex exponentials have magnitude one, the maximum and minimum values correspond to

$$(\Delta X_\theta)^2_{\text{max}} = \frac{1}{2}e^{2r}, (\Delta X_\theta)^2_{\text{min}} = \frac{1}{2}e^{-2r}. \tag{II.49}$$

The angles that lead to minimum and maximum variance are separated by $\pi/2$ and thus have the property that

$$(\Delta X_\theta)^2_{\text{max}}(\Delta X_\theta)^2_{\text{min}} = \frac{1}{4}. \tag{II.50}$$

Compared to the vacuum state and the coherent state, the variance along some quadratures are reduced relative to the vacuum level at the expense of enhanced fluctuations in other quadratures. This is the defining property of squeezing. The reduced variance in a particular quadrature enables stronger sensitivity in metrology when we are measuring a weak signal which is encoded in that particular quadrature. The property of having a variance which is reduced compared to the vacuum level is a quantum mechanical effect and cannot be reproduced by any classical random electromagnetic field.

So far, we have considered squeezed states generated from acting the squeezing operator $S(\zeta)$ on the vacuum state. These states have zero mean fields, but nonzero variances as we have just described. More generally, one can consider a larger family of squeezed states which are obtained by displacing squeezed vacuum states.

Thie can be done by defining a displaced squeezed state:

$$|\alpha, \zeta\rangle = D(\alpha)S(\zeta)\,|0\rangle\,. \tag{II.51}$$

Such a state has average properties given by the coherent displacement:

$$\langle\alpha, \zeta|a|\alpha, \zeta\rangle = \alpha, \qquad \langle\alpha, \zeta|a^\dagger|\alpha, \zeta\rangle = \alpha^*, \tag{II.52}$$

but retains the quadrature variance properties associated with the squeezed state. Such states are sometimes referred to as “bright squeezed states” since they have a nonzero average field. This is in contrast to squeezed vacuum states, which contain few photons on average.

D. (Optional) Derivation of Bogoliubov transformation on oscillator operators

In this optional section, we derive the Bogoliubov transformation generated by the single-mode squeezing operator. We begin with the definition:

$$S(\zeta) = \exp\left(\frac{1}{2}\zeta^* a^2 - \frac{1}{2}\zeta(a^\dagger)^2\right), \qquad \zeta = re^{i\phi}. \tag{II.53}$$

We define:

$$G \equiv \frac{1}{2}\zeta^* a^2 - \frac{1}{2}\zeta(a^\dagger)^2, \tag{II.54}$$

so that

$$S(\zeta) = e^G. \tag{II.55}$$

Our objective is to compute a transformation of the annihilation operator:

$$S^\dagger(\zeta)\, a\, S(\zeta) = e^{-G} a e^G. \tag{II.56}$$

We will use the BCH expansion,

$$e^{-G} a e^G = a + [a, G] + \frac{1}{2!}[[a, G], G] + \frac{1}{3!}[[[a, G], G], G] + \cdots \tag{II.57}$$

We first compute the needed commutators. Using $[a, a^\dagger] = 1$,

$$[a, a^2] = 0, \tag{II.58}$$

and

$$[a, (a^\dagger)^2] = [a, a^\dagger]a^\dagger + a^\dagger[a, a^\dagger] \tag{II.59}$$

$$= a^\dagger + a^\dagger \tag{II.60}$$

$$= 2a^\dagger. \tag{II.61}$$

Therefore,

$$[a, G] = \left[a, \frac{1}{2}\zeta^* a^2 - \frac{1}{2}\zeta(a^\dagger)^2\right] \tag{II.62}$$

$$= \frac{1}{2}\zeta^*[a, a^2] - \frac{1}{2}\zeta[a, (a^\dagger)^2] \tag{II.63}$$

$$= -\zeta a^\dagger. \tag{II.64}$$

Next,

$$[[a,G],G] = [-\zeta a^\dagger, G] \tag{II.65}$$

$$= -\zeta[a^\dagger, G]. \tag{II.66}$$

Now

$$[a^\dagger, (a^\dagger)^2] = 0, \tag{II.67}$$

and

$$[a^\dagger, a^2] = [a^\dagger, a]a + a[a^\dagger, a] \tag{II.68}$$

$$= -a - a \tag{II.69}$$

$$= -2a. \tag{II.70}$$

So

$$[a^\dagger, G] = \left[a^\dagger, \frac{1}{2}\zeta^* a^2 - \frac{1}{2}\zeta(a^\dagger)^2\right] \tag{II.71}$$

$$= \frac{1}{2}\zeta^*[a^\dagger, a^2] - \frac{1}{2}\zeta[a^\dagger, (a^\dagger)^2] \tag{II.72}$$

$$= -\zeta^* a. \tag{II.73}$$

Hence

$$[[a,G],G] = |\zeta|^2 a. \tag{II.74}$$

Continuing once more,

$$[[[a,G],G],G] = [|\zeta|^2 a, G] \tag{II.75}$$

$$= |\zeta|^2[a, G] \tag{II.76}$$

$$= -|\zeta|^2\zeta a^\dagger. \tag{II.77}$$

The pattern is now clear:

$$\text{odd nested commutators} \propto a^\dagger, \tag{II.78}$$

$$\text{even nested commutators} \propto a. \tag{II.79}$$

Substituting into the BCH series gives

$$e^{-G} a e^{G} = a - \zeta a^\dagger + \frac{|\zeta|^2}{2!}a - \frac{|\zeta|^2\zeta}{3!}a^\dagger + \frac{|\zeta|^4}{4!}a - \cdots \tag{II.80}$$

$$= \left(1 + \frac{|\zeta|^2}{2!} + \frac{|\zeta|^4}{4!} + \cdots\right)a - \left(\zeta + \frac{|\zeta|^2\zeta}{3!} + \frac{|\zeta|^4\zeta}{5!} + \cdots\right)a^\dagger. \tag{II.81}$$

Writing $\zeta = re^{i\phi}$ with $|\zeta| = r$, this becomes

$$e^{-G} a e^{G} = \left(1 + \frac{r^2}{2!} + \frac{r^4}{4!} + \cdots\right) a - e^{i\phi}\left(r + \frac{r^3}{3!} + \frac{r^5}{5!} + \cdots\right) a^\dagger \tag{II.82}$$

$$= a\cosh r - e^{i\phi} a^\dagger \sinh r. \tag{II.83}$$

Thus,

$$S^\dagger(\zeta)\, a\, S(\zeta) = a\cosh r - e^{i\phi} a^\dagger \sinh r. \tag{II.84}$$

Taking the Hermitian conjugate gives the companion relation

$$S^\dagger(\zeta)\, a^\dagger\, S(\zeta) = a^\dagger \cosh r - e^{-i\phi} a \sinh r. \tag{II.85}$$

This is the Bogoliubov transformation associated with single-mode squeezing. This mixing of creation and annihilation operators is typical of nonlinear optical interactions, as we will continue to see in later sections.

III. SCATTERING AND ATTENUATION OF LIGHT

Thus far, we have studied quantum states in isolation of all other influences. In other words, we constructed different quantum states of light (Fock states, coherent states, squeezed states) and analyzed their properties and the only dynamics we considered was their evolution under the Hamiltonian. In this section, we consider what happens when quantum light undergoes a beamsplitting operation. Although this sounds very specific, it turns out an analysis of a beamsplitter provides a general quantum-mechanically consistent framework for describing the attenuation of light, which, if done naively, violates the laws of quantum mechanics! The beamsplitter has some other beautiful and important properties. For example, it implements the displacement operation, and it also tells us that certain types of intricate quantum correlations such as squeezing are degraded by attenuation, which illustrates one of the central problems in the productive application of quantum optics. The beamsplitter analysis also immediately furnishes a theory of reflection and transmission.

In classical optics, it is common to model the behavior of a beamsplitter as a linear transformation between input and output fields. In particular, if $\alpha_{1,2}$ are the complex electric fields at the inputs, and $\beta_{1,2}$ are the complex electric fields at the outputs, then we can write[15]:

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} -r & it \\ it & -r \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}. \tag{III.1}$$

Here, r and t are complex reflection and transmission coefficients. The classical power between the two ports must be conserved. This puts a constraint on the values of the reflection and transmission coefficients. In particular, the matrix which transforms the fields must be unitary. This means

$$\begin{pmatrix} -r^* & -it^* \\ -it^* & -r^* \end{pmatrix} \begin{pmatrix} -r & it \\ it & -r \end{pmatrix} = \begin{pmatrix} |t|^2 + |r|^2 & -itr^* + it^*r \\ it^*r - itr^* & |t|^2 + |r|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{III.2}$$

The two independent equations that r and t must satisfy are thus:

$$|r|^2 + |t|^2 = 1 \tag{III.3}$$

$$rt^* - r^*t = 0. \tag{III.4}$$

The first constraint comes directly from power conservation, while the second can be seen a reciprocity condition. The exact values of r and t depend on the details of the physical implementation. If the beamsplitter consists of interfaces between different dielectric layers, then they can be found by solving a boundary condition problem for the fields in various layers.

Now we see that the beamsplitter induces a linear transformation on the classical fields. We may thus expect that this is also a valid *operator transformation* in the Heisenberg sense. In other words, we may think of a_1, a_2 as input field operators which get *transformed* by the beamsplitter to output field operators b_1 and b_2. Importantly, if the transformation is to be a valid Heisenberg transformation, then it must conserve all the relevant commutators. In other words, if $[a_i, a_j^\dagger] = \delta_{ij}$ then $[b_i, b_j^\dagger] = \delta_{ij}$. Let us denote the matrix relating (a_1, a_2) to (b_1, b_2) as S such that

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = S \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{III.5}$$

[15] There are multiple valid conventions for the S-matrix. I am using the one in Waves and Fields in Optoelectronics by Haus.

Then we may evaluate

$$[b_i, b_j^\dagger] = [S_{i1}a_1 + S_{i2}a_2, S_{j1}^* a_1^\dagger + S_{j2}^* a_2^\dagger] = S_{i1}S_{j1}^* + S_{i2}S_{j2}^* = \delta_{ij}. \tag{III.6}$$

In other words, the i and j rows must be orthonormal vectors, implying that S is unitary. Therefore, we may say that our beamsplitter, in the Heisenberg picture, implements a unitary matrix on the input operators and maps them to output operators. Let us now explore some of the consequences of this simple relationship.

A. Displacement operator

Let us consider a situation in which there is a quantum state incident on a first port described by annihilation operator a_1 while in the second port we have a coherent state $|\alpha\rangle$ incident on the second port described by operator a_2. For a strong coherent state in the second mode, we know that

$$a_2 = \alpha + \delta a_2, \tag{III.7}$$

where δa_2 represents the Heisenberg operator prior to displacement [16]. When taking expectation values involving a_2 they are now to be taken against the vacuum state (while they would have been taken against the coherent state had we simply worked with a_2). Now, let us consider the case when $|t|^2 \ll 1$ and $|t\alpha| \gg 1$ (meaning that the number of photons transmitted from port 2 is much larger than one). Then we may write

$$b_1 = -ra_1 + it(\alpha + \delta a_2). \tag{III.8}$$

Since $|r| \approx 1$ and $|t| \ll 1$, we see that in general $t\delta a_2$ will have negligible contribution to expectation values when $t \to 0$. Therefore, we may approximate

$$b_1 \approx \beta_1 - ra_1, \tag{III.9}$$

where $\beta = it\alpha_1$. Thus, we see that a_1 is displaced, up to an overall phase. In particular, taking $r = |r|e^{i\phi}$ and approximating $|r| \approx 1$, we have

$$b_1 \approx \beta_1 - e^{i\phi}a_1. \tag{III.10}$$

[16] The notation δ seems a bit odd at this point. We will make further use of it. But you should think of it as saying: the operator can be seen as mostly a c-number plus a quantum noise associated with the vacuum state.

We thus see that the beamsplitter acts as the displacement operator!

B. Beamsplitter as attenuation

Consider the case in which the first port has a state $|\psi\rangle$ incident on it and the second port has nothing incident on it. Quantum mechanics tells us that nothing does not really exist, and that we should think of the second port as having $|0\rangle$ incident on it. Let us compute how expectation values of various operators change as a result of this. Consider first the photon number operator

$$\langle b_1^\dagger b_1\rangle = \langle(-r^* a_1^\dagger - it^* a_2^\dagger)(-r a_1 + it a_2)\rangle = |r|^2\langle a_1^\dagger a_1\rangle. \qquad \text{(III.11)}$$

This is exactly what we intuitively expect. The energy incident in the first port is attenuated to a factor $|r|^2$. Quadrature operators in their expectation values also transform roughly in the way you expect

$$\langle X_{\theta,\text{out}}\rangle = \left\langle \frac{b_1 e^{-i\theta} + b_1^\dagger e^{i\theta}}{\sqrt{2}} \right\rangle = \frac{1}{\sqrt{2}}\left(-r a_1 e^{-i\theta} - r^* a_1^\dagger e^{i\theta}\right) = -|r|\langle X_{\theta-\phi,,\text{in}}\rangle, \quad \text{(III.12)}$$

where we have written $r = |r|e^{i\phi}$ and used the notations “in” and “out” to denote which operators are computed before and after the beamsplitter, respectively. There is a rotation of the quadrature which depends on the complex phase of the reflection coefficient.

Thus far, the fact that vacuum is incident on the second port of the beamsplitter has not worked its way in at all, and that is rather intuitive. Perhaps less intuitive is what happens to quadrature variances. Let us consider the expectation value of

$$\langle X_{\theta,\text{out}}^2\rangle = \frac{1}{2}\left\langle \left(b_1^2 e^{-2i\theta} + b_1^{\dagger 2} e^{2i\theta} + b_1 b_1^\dagger + b_1^\dagger b_1\right)\right\rangle. \qquad \text{(III.13)}$$

The first term is $-\frac{1}{2}r^2\langle a_1^2\rangle e^{-2i\theta}$, while the second term is $-\frac{1}{2}r^{*2}\langle a_1^{\dagger 2}\rangle e^{2i\theta}$. The third and fourth terms are most quickly evaluated by expressing their sum as $2\langle b_1^\dagger b_1\rangle + 1 = 2|r|^2\langle a_1^\dagger a_1\rangle + 1$.

We may now write the variance as

$$(\Delta X_{\theta,\text{out}})^2 = \frac{|r|^2}{2}\left(\langle a_1^2\rangle e^{-2i(\theta-\phi)} + \langle a_1^{\dagger 2}\rangle e^{2i(\theta-\phi)} + 2\langle a_1^\dagger a_1\rangle\right) + \frac{1}{2} - |r|^2\langle X_{\theta-\phi,\text{in}}\rangle^2. \qquad \text{(III.14)}$$

To get this into a cleaner form, we note that

$$X_{\theta-\phi,\mathrm{in}}^2 = \frac{a_1^2 e^{-2i(\theta-\phi)} + a_1^{\dagger 2} e^{+2i(\theta-\phi)} + 2a_1^\dagger a_1 + 1}{2}, \tag{III.15}$$

which allows us to write

$$(\Delta X_{\theta,\mathrm{out}})^2 = |r|^2 (\Delta X_{\theta-\phi,\mathrm{in}})^2 + \frac{1}{2}|t|^2, \tag{III.16}$$

where I have used $|r|^2 + |t|^2 = 1$. Naively, one might have expected that the attenuation by r would merely scale the quadrature variance by $|r|^2$ but we see that there is an other term which appears. This term makes the limit of $|r| \to 0$ physically sensible. For example, in the limit of complete attenuation, we expect to get vacuum which has variance of 1/2 in any quadrature. This is precisely what happens! If we did not include this term, we would have violated the uncertainty principle because we would have $\Delta X_\theta \Delta X_{\theta+\pi/2} < 1/2$.

An interesting consequence of this expression is the degradation of squeezing. Suppose without loss of generality that the reflection phase is $\phi = 0$ and we have light whose θ quadrature is squeezed such that $(\Delta X_{\theta,\mathrm{in}})^2 = \frac{1}{2}\epsilon$ with $\epsilon < 1$. Then

$$(\Delta X_{\theta,\mathrm{out}})^2 = \frac{1}{2}\left(|r|^2\epsilon + |t|^2\right) \tag{III.17}$$

The squeezing factor after the beamsplitter is

$$\epsilon + |t|^2(1-\epsilon) > \epsilon, \tag{III.18}$$

indicating that the squeezing is degraded. This turns out to be one of the most pernicious constraints towards using squeezed states for precision metrology. Any time we lose some of it, to anything, we are less able to exploit them for metrological gain! And loss is everywhere.

We have implemented attenuation using a beamsplitter. That said, it seems reasonable on physical grounds that absorptive loss could be seen in a similar way: by saying a portion of vacuum is mixed in. It turns out that a rigorous theory of absorptive loss does show that it can be treated by a beamsplitter approach, making this one of the most important models in quantum optics!

Before concluding, we also discuss the influence of a beamsplitter on photon number fluctuations. To evaluate it, we need to consider the second moment of the photon number at the output:

$$n_{\text{out}}^2 = b_1^\dagger b_1 b_1^\dagger b_1 = b_1^\dagger b_1^\dagger b_1 b_1 + b_1^\dagger b_1. \tag{III.19}$$

The expectation value of the quartic term is given by

$$\langle b_1^\dagger b_1^\dagger b_1 b_1 \rangle = |r|^4 \langle a_1^{\dagger 2} a_1^2 \rangle = |r|^4 \langle a_1^\dagger (a_1 a_1^\dagger - 1) a_1 \rangle. \tag{III.20}$$

Therefore

$$\langle b_1^\dagger b_1 b_1^\dagger b_1 \rangle = |r|^4 \langle n_{\text{in}}^2 \rangle + |t|^2 \langle n_{\text{in}} \rangle. \tag{III.21}$$

The variance then follows as

$$(\Delta n_{\text{out}})^2 = |r|^4 (\Delta n_{\text{in}})^2 + (|r|^2 - |r|^4) \langle n_{\text{in}} \rangle. \tag{III.22}$$

Noting that the fractional intensity output is $\ell = |r|^2$, we may write the result as

$$(\Delta n_{\text{out}})^2 = \ell^2 (\Delta n_{\text{in}})^2 + \ell(1-\ell) \langle n_{\text{in}} \rangle. \tag{III.23}$$

In the limit of large attenuation $\ell \to 0$, the variance becomes approximately $\ell \langle n_{\text{in}} \rangle = \langle n_{\text{out}} \rangle$ which corresponds to a Poissonian distribution of photons at the output.

This result (as well as the result for quadrature variances) depends critically on incorporating the second port. It is not entirely obvious from the derivation, but suppose we had assumed the mapping $b_1 = -r a_1$. In this case, the output commutator would be $[b_1, b_1^\dagger] = |r|^2 \neq 1$ for general r. Of course, this violates quantum mechanics (as it implies that we can have a product of quadrature variances below 1/2). But, taking this result, and doing the calculation *consistently*, one would find $\langle n_{\text{out}}^2 \rangle = |r|^4 \langle n_{\text{in}}^2 \rangle - |r|^4 \langle n_{\text{in}} \rangle + |r|^4 \langle n_{\text{in}} \rangle = |r|^4 \langle n_{\text{in}}^2 \rangle$. The last term arises from writing $b_1^\dagger b_1 b_1^\dagger b_1 = b_1^\dagger b_1^\dagger b_1 b_1 + |r|^2 b_1^\dagger b_1$ which comes from the use of the (incorrect!) commutator. The (incorrect) result which is obtained is what one would get if attenuation were *deterministic*. In other words, if loss mapped the photon number n to $|r|^2 n$ in a deterministic fashion, then the variance of the photon number would indeed just scale by $|r|^4$. Of course, this is not how scattering works. There is a *probability* of scattering $|r|^2$ and this is enough to recover the correct result. Additionally, you see from the quantization of light into photons that a deterministic scattering map of the type $n \to |r|^2 n$ is inconsistent with photons appearing in integer amounts. The intuition that is often stated in quantum optics is that the beamsplitter feeds vacuum fluctuations in from the empty port in a way that preserves commutation relations. Indeed you see without the second port, the commutator would break (at which point we should have not calculated anything further) and we would violate the uncertainty principle. You are encouraged to convince yourself that the result we obtained for quadrature variances also would have not been obtained if we had take $b_1 = -ra_1$.

C. Homodyne detection

We now turn to the question of how one can experimentally access the statistical properties of a state of light, particularly in the quantum regime. For example, we may wish to characterize quantities such as the quadrature variances of a weak quantum state, such as a squeezed vacuum state. Unlike photon number, these observables are phase-sensitive, and therefore require a measurement scheme capable of distinguishing between different quadratures (e.g., X vs. P). However, most photodetectors naturally measure intensity, not phase, suggesting that some form of interferometric technique is required. In addition, states such as squeezed vacuum may contain very few photons, making direct detection challenging. It is therefore desirable to have a method that allows us to probe such weak states using measurements involving large numbers of photons.

A powerful approach that addresses these challenges is *homodyne detection*. The basic idea is to interfere the quantum state we wish to characterize (the signal) with a strong reference beam of known phase, known as the *local oscillator*. By mixing the two fields on a beamsplitter, the statistical properties of the signal are effectively transferred onto the strong beam, enabling detection with high signal-to-noise. Furthermore, by varying the phase of the local oscillator, one can control the relative phase between the two fields, allowing phase-sensitive measurements that probe different quadratures of the signal state.

We now introduce the basic optical element underlying homodyne detection: a 50:50 beamsplitter. We denote the two input modes by a_1 and a_2, and the two output modes by b_1 and b_2. The beamsplitter mixes the input fields linearly, and in the convention used here the transformation is

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & i \\ i & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{III.24}$$

Equivalently, the output operators are

$$b_1 = \frac{1}{\sqrt{2}} \left(-a_1 + i a_2 \right), \tag{III.25}$$

$$b_2 = \frac{1}{\sqrt{2}} \left(i a_1 - a_2 \right). \tag{III.26}$$

Since this transformation is unitary, it preserves the bosonic commutation relations. For example,

$$[b_i, b_j^\dagger] = \delta_{ij}. \tag{III.27}$$

The corresponding output photon number operators are

$$n_{b_1} = b_1^\dagger b_1, \qquad n_{b_2} = b_2^\dagger b_2. \tag{III.28}$$

Using the beamsplitter relations, these become

$$n_{b_1} = \frac{1}{2}(-a_1^\dagger - i a_2^\dagger)(-a_1 + i a_2) \tag{III.29}$$

$$= \frac{1}{2} \left(a_1^\dagger a_1 - i a_1^\dagger a_2 + i a_2^\dagger a_1 + a_2^\dagger a_2 \right), \tag{III.30}$$

and

$$n_{b_2} = \frac{1}{2}(-i a_1^\dagger - a_2^\dagger)(i a_1 - a_2) \tag{III.31}$$

$$= \frac{1}{2} \left(a_1^\dagger a_1 + i a_1^\dagger a_2 - i a_2^\dagger a_1 + a_2^\dagger a_2 \right). \tag{III.32}$$

It is especially useful to consider the sum and difference of the two output photocurrents. The sum is

$$n_{b_1} + n_{b_2} = a_1^\dagger a_1 + a_2^\dagger a_2, \tag{III.33}$$

so the total photon number is conserved by the beamsplitter.

The difference is

$$n_{b_1} - n_{b_2} = i \left(a_2^\dagger a_1 - a_1^\dagger a_2 \right). \tag{III.34}$$

This operator will play a central role in homodyne detection. It shows that the difference signal at the two outputs is sensitive not simply to the intensity in one input mode alone, but to the interference between the two input fields.

At this stage we keep both input modes completely general. Later, one of these inputs will be taken to be a reference field (the local oscillator), and this will allow the output difference signal to probe specific field quadratures of the other mode.

We now assume that input mode a_2 is prepared in a coherent state and serves as the local oscillator:

$$a_2 \left|\alpha_{\mathrm{LO}}\right\rangle = \alpha_{\mathrm{LO}} \left|\alpha_{\mathrm{LO}}\right\rangle , \qquad \alpha_{\mathrm{LO}} = |\alpha_{\mathrm{LO}}|e^{i\theta}, \tag{III.35}$$

while a_1 is the signal mode and may be in an arbitrary quantum state.

From the previous section, the difference photocurrent operator is

$$I_- \equiv n_{b_1} - n_{b_2} = i\left(a_2^\dagger a_1 - a_1^\dagger a_2\right). \tag{III.36}$$

Taking the expectation value over the local-oscillator mode gives

$$\langle I_-\rangle = i\left\langle a_2^\dagger a_1 - a_1^\dagger a_2\right\rangle \tag{III.37}$$

$$= i\left(\alpha_{\mathrm{LO}}^* \langle a_1\rangle - \alpha_{\mathrm{LO}}\langle a_1^\dagger\rangle\right). \tag{III.38}$$

We define the signal quadrature in the standard form

$$X_\phi \equiv \frac{1}{\sqrt{2}}\left(a_1 e^{-i\phi} + a_1^\dagger e^{i\phi}\right). \tag{III.39}$$

Using $\alpha_{\mathrm{LO}} = |\alpha_{\mathrm{LO}}|e^{i\theta}$, we obtain

$$\langle I_-\rangle = i|\alpha_{\mathrm{LO}}|\left(e^{-i\theta}\langle a_1\rangle - e^{i\theta}\langle a_1^\dagger\rangle\right) \tag{III.40}$$

$$= \sqrt{2}\,|\alpha_{\mathrm{LO}}|\,\langle X_{\theta+\pi/2}\rangle. \tag{III.41}$$

Thus, the mean difference current directly measures the quadrature of the signal mode shifted by $\pi/2$ relative to the local-oscillator phase in this beamsplitter convention. Equivalently, one may write the measured quadrature as $X_{\theta-\pi/2}$ after a redefinition of the local-oscillator phase origin.

It is also useful to write this at the operator level. Since the local oscillator is in a coherent state, we may write

$$a_2 = \alpha_{\mathrm{LO}} + \delta a, \tag{III.42}$$

where δa describes the residual fluctuations of the local oscillator, with

$$\langle \delta a \rangle = 0, \qquad \langle \delta a^\dagger \delta a \rangle = 0, \qquad \langle \delta a \delta a^\dagger \rangle = 1. \tag{III.43}$$

Substituting into the difference current gives

$$I_- = i\left[(\alpha_{\text{LO}}^* + \delta a^\dagger)a_1 - a_1^\dagger(\alpha_{\text{LO}} + \delta a)\right] \tag{III.44}$$

$$= i\left(\alpha_{\text{LO}}^* a_1 - \alpha_{\text{LO}} a_1^\dagger\right) + i\left(\delta a^\dagger a_1 - a_1^\dagger \delta a\right) \tag{III.45}$$

$$= \sqrt{2}\,|\alpha_{\text{LO}}|\,X_{\theta+\pi/2} + i\left(\delta a^\dagger a_1 - a_1^\dagger \delta a\right). \tag{III.46}$$

Therefore, in the limit of a strong local oscillator, $|\alpha_{\text{LO}}| \gg 1$, the first term dominates and

$$I_- \approx \sqrt{2}\,|\alpha_{\text{LO}}|\,X_{\theta+\pi/2}. \tag{III.47}$$

This is the basic principle of homodyne detection: by changing the phase θ of the local oscillator, one selects which signal quadrature is measured.

We now compute the mean and variance of the difference current more explicitly. The mean is

$$\langle I_- \rangle = \sqrt{2}\,|\alpha_{\text{LO}}|\,\langle X_{\theta+\pi/2} \rangle. \tag{III.48}$$

For the variance, write

$$(\Delta I_-)^2 = \langle I_-^2 \rangle - \langle I_- \rangle^2. \tag{III.49}$$

Using

$$I_- = \sqrt{2}\,|\alpha_{\text{LO}}|\,X_{\theta+\pi/2} + i\left(\delta a^\dagger a_1 - a_1^\dagger \delta a\right), \tag{III.50}$$

and the fact that the signal and local-oscillator modes are independent, the cross terms vanish upon taking expectation values. Therefore,

$$(\Delta I_-)^2 = 2|\alpha_{\text{LO}}|^2 (\Delta X_{\theta+\pi/2})^2 + \left\langle \left[i(\delta a^\dagger a_1 - a_1^\dagger \delta a)\right]^2 \right\rangle. \tag{III.51}$$

The second term is

$$\left\langle \left[i(\delta a^\dagger a_1 - a_1^\dagger \delta a)\right]^2 \right\rangle = \langle a_1^\dagger a_1 \rangle \langle \delta a \delta a^\dagger \rangle \tag{III.52}$$

$$= \langle a_1^\dagger a_1 \rangle, \tag{III.53}$$

since all other terms vanish for a coherent-state fluctuation mode d.

Thus the exact variance is

$$(\Delta I_-)^2 = 2|\alpha_{\rm LO}|^2(\Delta X_{\theta+\pi/2})^2 + \langle a_1^\dagger a_1\rangle. \tag{III.54}$$

In the strong-local-oscillator limit, the first term dominates, and one obtains

$$(\Delta I_-)^2 \approx 2|\alpha_{\rm LO}|^2(\Delta X_{\theta+\pi/2})^2. \tag{III.55}$$

So both the mean and the variance of the difference photocurrent are directly proportional to the corresponding mean and variance of the selected signal quadrature. This is why homodyne detection provides a direct experimental probe of quadrature statistics, including squeezing.

D. Sub shot-noise interferometry

As we know from elementary courses on waves, interferometry gives us a tool to measure small changes in length by translating them to phase shifts that manifest as changes in the intensity of light. We will perform a quantum analysis of an interferometer and show that the resolvable phase shift is limited by vacuum fluctuations. Then we show that by using squeezed states, the sensitivity can be enhanced.

Let us consider a Mach-Zehnder interferometer. Light enters one port of a 50/50 beamsplitter with the other port being empty. The two arms undergo a relative phase shift ϕ and are then recombined by another 50/50 beamsplitter. As we know quantum mechanically, the empty port corresponds to vacuum being sent in, and we must keep track of this. Let the two input arms have lowering operators a_1, a_2. After passing through the beamsplitter, they are mapped to

$$b_1 = \frac{1}{\sqrt{2}}\left(-a_1 + ia_2\right), b_2 = \frac{1}{\sqrt{2}}\left(ia_1 - a_2\right). \tag{III.56}$$

Let us assume it is the second arm that experiences an additional phase shift, so that the operators just before the second 50/50 beamsplitter may be written as

$$c_1 = \frac{1}{\sqrt{2}}\left(-a_1 + ia_2\right), c_2 = \frac{1}{\sqrt{2}}\left(ia_1 - a_2\right)e^{-i\theta}. \tag{III.57}$$

At the second 50/50 beamsplitter, the output operators in the first output port may be written as

$$d_1 = \frac{1}{\sqrt{2}}\left(-c_1 + ic_2\right) = \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\left(-a_1 + ia_2\right) + i\frac{1}{\sqrt{2}}\left(ia_1 - a_2\right)e^{-i\theta}\right), \tag{III.58}$$

which may be simplified as

$$d_1 = \frac{1}{2}\left(a_1(1 - e^{-i\theta}) - ia_2(1 + e^{-i\theta})\right). \tag{III.59}$$

The second output port operator may be written as

$$d_2 = \frac{1}{\sqrt{2}}(ic_1 - c_2) = \frac{1}{\sqrt{2}}\left(i\frac{1}{\sqrt{2}}(-a_1 + ia_2) - \frac{1}{\sqrt{2}}(ia_1 - a_2)\, e^{-i\theta}\right), \tag{III.60}$$

which may be similarly simplified as

$$d_2 = \frac{1}{2}\left(-ia_1(1 + e^{-i\theta}) + a_2(e^{-i\theta} - 1)\right). \tag{III.61}$$

It is helpful to express the overall transformation in terms of an S matrix, which is found to be:

$$S = \frac{1}{2}\begin{pmatrix} (1 - e^{-i\theta}) & -i(1 + e^{-i\theta}) \\ -i(1 + e^{-i\theta}) & (e^{-i\theta} - 1) \end{pmatrix} = ie^{-\frac{i\theta}{2}}\begin{pmatrix} \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \\ -\cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \end{pmatrix}, \tag{III.62}$$

It is easy to see from the trigonometric form that this transformation is unitary.

The photon numbers associated with the output ports may be written as

$$d_1^\dagger d_1 = \sin^2\left(\frac{\theta}{2}\right) a_1^\dagger a_1 + \cos^2\left(\frac{\theta}{2}\right) a_2^\dagger a_2 - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)(a_1^\dagger a_2 + a_2^\dagger a_1) \tag{III.63}$$

and

$$d_2^\dagger d_2 = \cos^2\left(\frac{\theta}{2}\right) a_1^\dagger a_1 + \sin^2\left(\frac{\theta}{2}\right) a_2^\dagger a_2 + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)(a_1^\dagger a_2 + a_2^\dagger a_1). \tag{III.64}$$

Let us now consider the configuration of this interferometer a bit more carefully now. In practice, the phase shift we are trying to measure is small. We would like our signal to be zero when there is no phase shift and to increase from zero when the source of the phase shift is present. This leads to two choices in operating this interferometer. The first is that we will "bias" the interferometer so that in the absence of our source phase shift, there is a phase-shift between the arms of $\pi/2$, so the overall phase shift is $\pi/2 + \theta$. Thus we make the mapping $\theta \to \frac{\pi}{2} + \theta$ in the above expressions. When the phase shift is exactly $\pi/2$ ($\theta = 0$) then we see from the above photon-number expressions that classically, when the a_2 port is empty, the photon number in each arm is identical. Since we'd like the signal to increase from zero, we subtract the two photon numbers. Therefore, we define our signal to be

$$S = d_1^\dagger d_1 - d_2^\dagger d_2. \tag{III.65}$$

Since θ is small, we will expand around $\theta = 0$, in which case the signal is given by (to linear order in θ)

$$S = \theta(a_1^\dagger a_1 - a_2^\dagger a_2) - (a_1^\dagger a_2 + a_2^\dagger a_1) \equiv \theta(a_1^\dagger a_1 - a_2^\dagger a_2) + C. \tag{III.66}$$

The mean value of the signal in the empty port configuration is

$$\langle S \rangle = \theta \langle a_1^\dagger a_1 \rangle. \tag{III.67}$$

The variance of the signal can be found quickly as follows. The expectation value of S^2 has a term which is quadratic in θ, a term which is C^2, and a cross term. The cross term makes no contribution because the expectation values of operators linear and cubic in a_2 vanish. The θ^2 term has a variance of $\theta^2 \Delta(a_1^\dagger a_1)^2$ which we will find is small[17]. Taking the input field to be a coherent state $|\alpha\rangle$ with α real, X may be approximated via

$$C \approx -\alpha(a_2 + a_2^\dagger) = -\sqrt{2}\alpha X_0, \tag{III.68}$$

where X_0 is a quadrature operator corresponding to a_2. Therefore, the total variance may be written as

$$(\Delta S)^2 = \theta^2 \alpha^2 + 2\alpha^2 (\Delta X_0)^2, \tag{III.69}$$

which for the vacuum state may be written as (using $(\Delta X_0)^2 = 1/2$)

$$(\Delta S)^2 = (1 + \theta^2)\alpha^2 \approx \alpha^2. \tag{III.70}$$

The variance comes entirely from vacuum fluctuations of the empty port, amplified by the mean signal! The measure of the signal-to-noise ratio of the measurement is given by the relative variance

$$\frac{\Delta S}{\langle S \rangle} = \frac{1}{\theta} \frac{1}{\sqrt{\langle n \rangle}}. \tag{III.71}$$

This tells us that the weaker the phase shift, the harder it is to see, but that by having many photons, we can resolve it more easily. You can think of each photon as sampling the phase corresponding to n trials of an experiment giving the expected

[17] This follows from the two input ports having no correlations, in which case the variance of the difference of photon numbers is simply the sum of the variances plus twice the covariance.

$\sqrt{n}$ behavior. This $1/\sqrt{n}$ scaling is called the shot-noise limit and is the variance imposed by simply having the empty port present.

The variance can be reduced by sending a squeezed vacuum state into the empty port. We shall assume that $\sinh^2 r \ll |\alpha|^2$: that the number of photons in the squeezed vacuum field is far less than the number of photons in the coherent laser field (this in practice is more than well-satisfied). Repeating the calculation for squeezed input under these approximations, the mean signal is unchanged. The variance however becomes $2\alpha^2(\Delta X_0)^2$ (neglecting θ^2). Assuming that our squeezed vacuum is squeezed in the right quadrature, the variance becomes $\alpha^2 e^{-2r}$ and our relative variance becomes

$$\frac{\Delta S}{\langle S\rangle} = \frac{1}{\theta}\frac{e^{-r}}{\sqrt{\langle n\rangle}}, \tag{III.72}$$

corresponding to an enhancement in the signal-to-noise ratio for the same intensity. This is a defining property of squeezed states and is what is currently used at advanced LIGO to detect gravitational waves. Gravitational waves correspond to propagating distortions of the spacetime metric which lead to changes in the distances between points in space. These changes can be measured interferometrically by detecting the corresponding phase shift of light. The main problem however is that these phase shifts are extraordinarily small. Therefore even with high intensity inputs into the interferometer, the signal-to-noise ratio is insufficient to resolve many different gravitational wave events. While one can always crank up the intensity to get a better signal-to-noise ratio, in practice the interferometers have a maximum laser intensity that they can handle. The injection of squeezed vacuum into the empty port gives lower variance for the same intensity, mimicking an intensity enhanced by e^r.

The principle of using squeezed vacuum can be used for any system which is limited by the shot noise of light and these ideas are now being explored in microscopy, biological imaging, and magnetometry, among other fields.